In the Name of God

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A CONSTRAINED SOLID TSP IN FUZZY ENVIRONMENT: TWO HEURISTIC APPROACHES

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Abstract. A solid travelling salesman problem (STSP) is a travelling salesman problem (TSP) where the salesman visits all the cities only once in his tour using different conveyances to travel from one city to another. Costs and environmental effect factors for travelling between the cities using different conveyances are different. Goal of the problem is to find a complete tour with minimum cost that damages the environment least. An ant colony optimization (ACO) algorithm is developed to solve the problem. Performance of the algorithm for the problem is compared with another soft computing algorithm, Genetic Algorithm(GA). Problems are solved with crisp as well as fuzzy costs. For fuzzy cost and environmental effect factors, cost function as well as environment constraints become fuzzy. As optimization of a fuzzy objective function is not well defined, fuzzy possibility approach is used to get optimal decision. To test the efficiency of the algorithm, the problem is solved considering only one conveyance facility ignoring the environmental effect constraint, i.e., a classical two dimensional TSP (taking standard data sets from TSPLIB for solving the problem). Different numerical examples are used for illustration.

1. Introduction

The TSP consists of a complete graph of N vertices whose edges represent distances/costs between the nodes associated with it. The goal is to find the shortest possible tour through the vertices (nodes/cities) so that each vertex is visited exactly once except the starting vertex. Tour ends at starting vertex. This problem is known to be NP-hard and cannot be solved exactly in polynomial time [17, 18]. Different types of TSPs have been solved by the researchers during last two decades. Among these, TSP with time windows [12], stochastic TSP [3, 19], double TSP [32], asymmetric TSP [23], TSP with precedence constraints [26], etc are worth mentioning. In TSP with precedence constraints, there exists an order in which the vertices should be visited. In asymmetric TSP, cost of travelling from vertex (node/city) v_i to v_j is not equal to the cost of travelling from vertex v_j to v_i. In stochastic TSP, each vertex is visited with a given probability and goal is to minimize the expected distance/cost of a priori tour. In TSP with time windows, each vertex is visited within a specified time windows.

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In all the above literature it is implicitly assumed that travelling cost from one node to another is fixed, i.e., crisp in nature. But travelling cost from one node to another depends on the conveyance used for travelling. Also it varies slightly depending on the availability of the conveyance, condition of the road, etc., though its value normally lies in an interval. Due to this reason it is better to model the costs of a TSP as fuzzy numbers. It is less error prone as these estimations are based on experts opinion. Also problem should be modelled in such a way that salesman can visit one city to another using different conveyances. Though it is normally practised by salesmen, no researcher has considered TSP with different conveyance facilities i.e. solid TSP. Again environment is differently adversely effected in travelling through different conveyances. So goal should be such that the tour should damage the environment least, i.e., it should be maximum echo-friendly. Till now, none has considered this constraint even in classical TSP. But now-a-days, this environment friendly travel, that is tour using gas filed vehicles, etc, is given maximum importance.

Removing the above shortcomings, here a TSP is considered where salesman can use different conveyances to travel from one city to another. Cost and environmental friendliness for travelling using different conveyances from one city to another are different. The salesman should maintain a minimum environmental friendliness to damage the environment in his tour up to the maximum permitted limit. Crisp as well as fuzzy costs and echo-friendliness coefficients are used in different problems. A fuzzy possibility based approach is used to make optimal decision for fuzzy costs and environmental effect factors (in positive sense). As three dimensional cost and echo-friendliness coefficients are required to represent the problem, the problem is named as Solid TSP.

In the existing literature besides exact methods, meta-heuristics, local search and hybrid algorithms of optimization and searching approaches are applied to solve TSPs. Exact methods include cutting plane, LP relaxation [6], branch and bound [30], branch and cut [31], dynamic programming [11] etc. However very small size problems can be solved by exact methods. On the other hand large size problems are solved using heuristics like Simulated Annealing [5], Local Search [14], Hybrid Algorithm [12], Tabu search [15], Genetic Algorithm (GA) [26, 28], Ant Colony Optimization (ACO)[2]. Heuristic algorithms establish their efficiency for solving different real life problems in imprecise environment[16, 20, 27, 35]. The present problem under investigation is more complicated for its imprecise costs and constraints on environmental effect. Fuzziness of the cost and environmental effect leads to fuzzy total-cost with a fuzzy constraint. As optimization of fuzzy objective is not well defined and hence it is very difficult to find optimal paths for the stated problem. Due to this complexity, a fuzzy possibility/necessity based approach is proposed to transfer fuzzy objective into an equivalent crisp objective (see section-3). An ACO is developed to find optimal path of such a realistic solid TSP. A GA with cyclic crossover, two-point mutation is also used to find optimal paths. Results through two approaches are compared to check the validity of ACO for the problem. Finally problem is illustrated with numerical examples. The novelty of the paper is the introduction of solid TSP for the first time in the field of TSP and

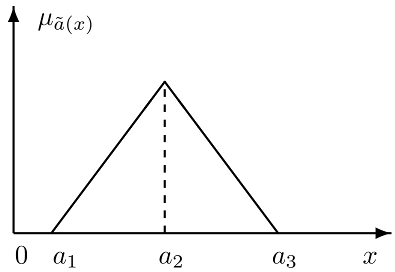

FIGURE 1. Triangular Fuzzy Number $a = (a_1, a_2, a_3)$

the consideration of the environmental effect during the tour. The problem is also solved considering only one conveyance facility ignoring environmental effect(i.e. classical TSP) using standard data set from the literature (taking data set from TSPLIB), which establishes that the proposed approach is sufficiently powerful to solve the considered general problem.

2. Mathematical Prerequisite

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively. Then according to [9, 36],

$$pos(\tilde{a} * \tilde{b}) = sup\{min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \Re, x * y\} \quad (1)$$

where the abbreviation pos represents possibility, $*$ is any one of the relations $>, <, =, \leq, \geq$ and $\Re$ represents set of real numbers.

$$nes(\tilde{a} * \tilde{b}) = 1 - pos(\overline{\tilde{a} * \tilde{b}}) \quad (2)$$

where the abbreviation nes represents necessity.

If $\tilde{a}$, $\tilde{b} \subseteq \Re$ and $\tilde{c} = f(\tilde{a}, \tilde{b})$ where $f : \Re \times \Re \to \Re$ is a binary operation then membership function $\mu_{\tilde{c}}$ of $\tilde{c}$ is defined as in [37]

$$\text{For each } z \in \Re,\ \mu_{\tilde{c}}(z) = sup\{min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \Re \text{ and } z = f(x, y)\} \quad (3)$$

Triangular fuzzy number(TFN): A TFN $\tilde{a} = (a_1, a_2, a_3)$ (cf. Fig-1) has three parameters a_1, a_2, a_3 where $a_1 < a_2 < a_3$ and is characterized by the membership function $\mu_{\tilde{a}}$, given by

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{for } a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2} & \text{for } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

According to the above definitions following lemmas can easily be derived (see e.g. [21, 22]).

Lemma 2.1. *If $\tilde{a} = (a_1, a_2, a_3)$ be a TFN with $0 < a_1$ and b a crisp number, then $pos(\tilde{a} < b) \geq \alpha$ iff $\dfrac{b - a_1}{a_2 - a_1} \geq \alpha$.*

Lemma 2.2. *If* $\tilde{a} = (a_1, a_2, a_3)$ *be a TFN with* $0 < a_1$ *and* b *a crisp number then* $nes(\tilde{a} < b) \geq \alpha$ *iff* $\frac{a_3 - b}{a_3 - a_2} \leq 1 - \alpha$.

Lemma 2.3. *If* $\tilde{a} = (a_1, a_2, a_3)$ *and* $\tilde{b} = (b_1, b_2, b_3)$ *be TFNs with* $0 < a_1$ *and* $0 < b_1$, *then* $pos(\tilde{a} > \tilde{b}) \geq \alpha$ *iff* $\frac{a_3 - b_1}{a_3 - a_2 + b_2 - b_1} \geq \alpha$.

Lemma 2.4. *If* $\tilde{a} = (a_1, a_2, a_3)$ *and* $\tilde{b} = (b_1, b_2, b_3)$ *be TFNs with* $0 < a_1$ *and* $0 < b_1$ *then* $nes(\tilde{a} > \tilde{b}) \geq \alpha$ *iff* $\frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2} \leq 1 - \alpha$.

3. Problem Definition

3.1. **General TSP with Environmental Effect Constraints.** In a classical two-dimensional travelling salesman problem, a salesman has to travel N cities using minimum cost. In his tour salesman starts from a city, visits all the cities exactly once and comes to the starting city using minimum cost. Here a conveyances echo-friendliness in travelling from one city to another is considered. The salesman should choice such a path in which a minimum environmental damage is ensured, on the other hand, maximum environment conservation is maintained. Let $c(i, j)$ be the cost for travelling from i-th city to j-th city and $s(i, j)$ be the travel comfort in travelling from i-th city to j-th city. Then the problem can be mathematically formulated as:

$$\left\{\begin{array}{rl} \text{Minimize} & Z = \sum_{i=1}^{N}\sum_{j=1}^{N} t_{ij} c(i,j) \\ \text{subject to} & \sum_{i=1}^{N} t_{ij} = 1 \text{ for } j = 1, 2, ..., N \\ & \sum_{j=1}^{N} t_{ij} = 1 \text{ for } i = 1, 2, ..., N \\ & \sum_{i=1}^{N}\sum_{j=1}^{N} t_{ij} s(i,j) \geq s_{min} \end{array}\right. \tag{5}$$

where $t_{ij} = 1$ if the salesman travels from city-i to city-j, otherwise $t_{ij} = 0$ and s_{min} is the minimum environmental effect level that should be maintained by the salesman. Let$(x_1,\ x_2,\ ...,x_N,x_1)$ be a complete tour of a salesman, where $x_i \in \{1, 2, ..N\}$, for $i = 1, 2, ..., N$ and all x_i are distinct. Then the above problem reduces to

$$\left\{\begin{array}{l} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \\ \text{to minimize} \quad Z = \sum_{i=1}^{N-1} c(x_i, x_{i+1}) + c(x_N, x_1) \\ \text{subject to} \quad \sum_{i=1}^{N-1} s(x_i, x_{i+1}) + s(x_N, x_1) \geq s_{min} \end{array}\right. \tag{6}$$

3.2. **Classical Fuzzy TSP with Environmental Effect Constraints.** In the above problem if costs and environmental effect factors be fuzzy numbers, i.e, $\tilde{c}(i, j)$

and $\tilde{s}(i,j)$ respectively, and environmental effect limit s_{min} and also fuzzy number $\tilde{s}_{min}$, then the above problem reduces to:

$$\begin{cases} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \\ \text{to minimize} \quad \tilde{Z} = \sum_{i=1}^{N-1} \tilde{c}(x_i, x_{i+1}) + \tilde{c}(x_N, x_1) \\ \text{subject to} \quad \sum_{i=1}^{N-1} \tilde{s}(x_i, x_{i+1}) + \tilde{s}(x_N, x_1) \geq \tilde{s}_{min} \end{cases} \quad (7)$$

As minimization of fuzzy objective is not well defined, the problem can be equivalently treated as

$$\begin{cases} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \text{ to minimize} \quad F \\ \text{subject to} \quad \sum_{i=1}^{N-1} \tilde{c}(x_i, x_{i+1}) + \tilde{c}(x_N, x_1) < F \\ \sum_{i=1}^{N-1} \tilde{s}(x_i, x_{i+1}) + \tilde{s}(x_N, x_1) \geq \tilde{s}_{min} \end{cases} \quad (8)$$

where F is any crisp parameter. It is clear that minimization of F implies minimization of fuzzy objective $\tilde{Z}$. For this reason the above maintained approach is used to treat fuzzy objective $\tilde{Z}$.

Again minimization of fuzzy constraints are not well defined. So following section-2 the above problem can be rewritten in optimistic and pessimistic sense by (9) and (10) respectively.

$$\begin{cases} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \text{ to minimize} \quad F \\ \text{subject to} \quad Pos(\sum_{i=1}^{N-1} \tilde{c}(x_i, x_{i+1}) + \tilde{c}(x_N, x_1) < F) \geq \alpha_1 \\ Pos(\sum_{i=1}^{N-1} \tilde{s}(x_i, x_{i+1}) + \tilde{s}(x_N, x_1) \geq \tilde{s}_{min}) \geq \beta_1 \end{cases} \quad (9)$$

$$\begin{cases} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \text{ to minimize} \quad F \\ \text{subject to} \quad Nes(\sum_{i=1}^{N-1} \tilde{c}(x_i, x_{i+1}) + \tilde{c}(x_N, x_1) < F) \geq \alpha_2 \\ Nes(\sum_{i=1}^{N-1} \tilde{s}(x_i, x_{i+1}) + \tilde{s}(x_N, x_1) \geq \tilde{s}_{min}) \geq \beta_2 \end{cases} \quad (10)$$

where α_1, β_1 and α_2, β_2 of equation (9) and (10) are predefined levels of possibility and necessity respectively which are entirely determined by the salesman. Their significance is discussed at the end of this subsection.

If we consider the fuzzy numbers as TFNs, $\tilde{c}(i,j) = (c(i,j)_1,\ c(i,j)_2,\ c(i,j)_3)$, $\tilde{s}(i,j) = (s(i,j)_1,\ s(i,j)_2,\ s(i,j)_3)$ and $\tilde{s}_{min} = (s_1, s_2, s_3)$, following section-2 the above problems (9) and (10) reduces to (11) and (12) respectively.

$$\begin{cases} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \text{ to minimize} \quad F \\ \text{subject to} \quad \dfrac{F - C_1}{C_2 - C_1} \geq \alpha_1 \\ \dfrac{S_3 - s_1}{S_3 - S_2 + s_2 - s_1} \geq \beta_1 \end{cases} \quad (11)$$

$$\text{where } C_j = \sum_{i=1}^{N-1} c(x_i, x_{i+1})_j + c(x_N, x_1)_j, \ j = 1, 2, 3.$$

$$\text{and } S_j = \sum_{i=1}^{N-1} s(x_i, x_{i+1})_j + s(x_N, x_1)_j, \ j = 1, 2, 3.$$

$$\begin{cases} \text{Determine a complete tour } (x_1, x_2, ..., x_N, x_1) \text{ to minimize } F \\ \text{subject to } \dfrac{C_3 - F}{C_3 - C_2} \leq 1 - \alpha_2 \\ \dfrac{s_3 - S_1}{S_2 - S_1 + s_3 - s_2} \leq 1 - \beta_2 \end{cases} \quad (12)$$

Which are equivalent to (13) and (14) respectively.

$$\begin{cases} \text{Determine a complete tour } (x_1, x_2, ..., x_N, x_1) \\ \text{to minimize } C_1 + \alpha_1 (C_2 - C_1) \\ \text{subject to } \dfrac{S_3 - s_1}{S_3 - S_2 + s_2 - s_1} \geq \beta_1 \end{cases} \quad (13)$$

$$\begin{cases} \text{Determine a complete tour } (x_1, x_2, ..., x_N, x_1) \\ \text{to minimize } C_3 - (1 - \alpha_2)(C_3 - C_2) \\ \text{subject to } \dfrac{s_3 - S_1}{S_2 - S_1 + s_3 - s_2} \leq 1 - \beta_2 \end{cases} \quad (14)$$

If the salesman is most optimistic then he will choice value of α_1, β_1 nearly 0 and in that case minimum possible cost function (C_1) is minimized assuming maximum possible environmental echo-friendliness of the tour (S_3) reaches the minimum possible environmental effect requirement (s_1). On the other hand if the salesman is least optimistic then he/she will choice values of α_1, $\beta_1$nearly 1 and in that case most feasible cost function (C_2) is minimized assuming most feasible environmental friendliness or travel comfort of the tour (S_2) reaches the most feasible environmental effect requirement (s_2). Pessimistic salesman will go for necessity approach. If he is most pessimistic, he will choice values of α_2, β_2 nearly 1 and in that case maximum possible cost function (C_3) is minimized assuming minimum possible environmental effect of the tour (S_1) reaches the maximum possible environmental effect requirement (s_3). On the other hand if the salesman is least pessimistic then he/she will choice value of α_2, β_2 nearly 0 and in that case most feasible cost function (C_2) is minimized assuming most feasible environmental effect of the tour (S_2) reaches the most feasible environmental effect requirement (s_2).

3.3. **Solid TSP with Environmental Effect Constraints.** In a solid travelling salesman problem, a salesman has to travel N cities by choosing any one of the M conveyances available using minimum cost. In his tour salesman starts from a city, visits all the cities exactly once using suitable conveyances available at the cities and comes to the starting city using minimum cost. Environmental factor in travelling from one city to another using different conveyances are different. The salesman should choice such a path and conveyances such that a minimum environmental effect is maintained. Let $c(i, j, k)$ be the cost of travelling from i-th city to j-th city using k-th type conveyance and $s(i, j, k)$ be the environmental effect in travelling from i-th city to j-th using k-th type conveyance. Then the salesman has to determine a complete tour $(x_1, x_2, ..., x_N, x_1)$ in which a particular or different combinations of conveyance types $(v_1, v_2, ..., v_N)$ is to be used for the tour, where $x_i \in \{1, 2, ..N\}$ for $i = 1, 2, ..., N$, $v_i \in \{1, 2, ..M\}$ for $i = 1, 2, ..., N$ and all x_i's are distinct. Then the problem can be mathematically formulated as:

$$\left\{\begin{array}{l} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \\ \text{and corresponding conveyance types} \;\; (v_1, v_2, ..., v_N) \\ \text{to minimize} \quad Z = \sum_{i=1}^{N-1} c(x_i, x_{i+1}, v_i) + c(x_N, x_1, v_N) \\ \text{subject to} \quad \sum_{i=1}^{N-1} s(x_i, x_{i+1}, v_i) + s(x_N, x_1, v_N) \geq s_{min} \end{array}\right. \tag{15}$$

where s_{min} is the minimum environment friendliness travel comfort that should be maintained by the salesman.

3.4. **Solid TSP with Fuzzy Costs and Environmental Effect Constraints.** In the above problem if costs and environmental effect factors are fuzzy numbers, i.e, $\tilde{c}(i, j, k)$ and $\tilde{s}(i, j, k)$ respectively and environmental effect limit s_{min} also fuzzy number $\tilde{s}_{min}$, the above problem reduces to:

$$\left\{\begin{array}{l} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \\ \text{and corresponding conveyance} \quad (v_1, v_2, ..., v_N) \\ \text{to minimize} \quad Z = \sum_{i=1}^{N-1} \tilde{c}(x_i, x_{i+1}, v_i) + \tilde{c}(x_N, x_1, v_N) \\ \text{subject to} \quad \sum_{i=1}^{N-1} \tilde{s}(x_i, x_{i+1}, v_i) + \tilde{s}(x_N, x_1, v_N) \geq \tilde{s}_{min} \end{array}\right. \tag{16}$$

As minimization of fuzzy objective as well as fuzzy constraints are not well defined following section-2 (as discussed in section-3.2) the above problem can be rewritten in optimistic and pessimistic sense by (17) and (18) respectively as

$$\left\{\begin{array}{l} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \\ \text{and corresponding conveyance} \;\; (v_1, v_2, ..., v_N) \;\text{to minimize} \quad Z \\ \text{subject to} \quad Pos(\sum_{i=1}^{N-1} \tilde{c}(x_i, x_{i+1}, v_i) + \tilde{c}(x_N, x_1, v_N) < Z) \geq \alpha_3 \\ \qquad Pos(\sum_{i=1}^{N-1} \tilde{s}(x_i, x_{i+1}, v_i) + \tilde{s}(x_N, x_1, v_N) \geq \tilde{s}_{min}) \geq \beta_3 \end{array}\right. \tag{17}$$

$$\left\{\begin{array}{l} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \\ \text{and corresponding conveyance} \;\; (v_1, v_2, ..., v_N) \;\text{to minimize} \quad Z \\ \text{subject to} \quad Nes(\sum_{i=1}^{N-1} \tilde{c}(x_i, x_{i+1}, v_i) + \tilde{c}(x_N, x_1, v_N) < Z) \geq \alpha_4 \\ \qquad Nes(\sum_{i=1}^{N-1} \tilde{s}(x_i, x_{i+1}, v_i) + \tilde{s}(x_N, x_1, v_N) \geq \tilde{s}_{min}) \geq \beta_4 \end{array}\right. \tag{18}$$

where α_3, β_3 and α_4, β_4 are predefined levels of possibility and necessity respectively which are entirely determined by the salesman. The meaning of α_3, β_3, α_4 and β_4 are the same as α_2, β_2, α_3 and β_3 of section-3.2, respectively.
If we consider the fuzzy numbers as TFNs, $\tilde{c}(i,j,k) = (c(i,j,k)_1, c(i,j,k)_2, c(i,j,k)_3)$, $\tilde{s}(i,j,k) = (s(i,j,k)_1, s(i,j,k)_2, s(i,j,k)_3)$ and $\tilde{s}_{min} = (s_1, s_2, s_3)$, then following section-2 the above problems (17), (18) reduces to (19), (20) respectively as below:

$$\left\{\begin{array}{c} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \\ \text{and corresponding conveyance types } (v_1, v_2, ..., v_N) \text{ to minimize} \quad Z \\ \text{subject to} \quad \dfrac{Z - F_1}{F_2 - F_1} \geq \alpha_3 \\ \dfrac{G_3 - s_1}{G_3 - G_2 + s_2 - s_1} \geq \beta_3 \end{array}\right. \tag{19}$$

$$\text{where} \quad F_j = \sum_{i=1}^{N-1} c(x_i, x_{i+1}, v_i)_j + c(x_N, x_1, v_N)_j, \; j = 1, 2, 3.$$

$$\text{and} \quad G_j = \sum_{i=1}^{N-1} s(x_i, x_{i+1}, v_i)_j + s(x_N, x_1, v_N)_j, \; j = 1, 2, 3.$$

$$\left\{\begin{array}{c} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \\ \text{and corresponding conveyance} \quad (v_1, v_2, ..., v_N) \text{ to minimize} \quad Z \\ \text{subject to} \quad \dfrac{F_3 - Z}{F_3 - F_2} \leq 1 - \alpha_4 \\ \dfrac{s_3 - G_1}{G_2 - G_1 + s_3 - s_2} \leq 1 - \beta_4 \end{array}\right. \tag{20}$$

which are equivalent to (21) and (22) respectively.

$$\left\{\begin{array}{c} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \\ \text{and corresponding conveyance} \; (v_1, v_2, ..., v_N) \\ \text{to minimize} \quad F_1 + \alpha_3(F_2 - F_1) \\ \text{subject to} \quad \dfrac{G_3 - s_1}{G_3 - G_2 + s_2 - s_1} \geq \beta_3 \end{array}\right. \tag{21}$$

$$\left\{\begin{array}{c} \text{Determine a complete tour} \quad (x_1, x_2, ..., x_N, x_1) \\ \text{and corresponding conveyance} \; (v_1, v_2, ..., v_N) \\ \text{to minimize} \quad F_3 - (1 - \alpha_4)(F_3 - F_2) \\ \text{subject to} \quad \dfrac{s_3 - G_1}{G_2 - G_1 + s_3 - s_2} \leq 1 - \beta_4 \end{array}\right. \tag{22}$$

4. **Ant Colony Optimization Algorithm**

Observing the foraging behavior of ants that their ability in finding the shortest path between their nest and a food sources, Dorigo [7] first developed an algorithmic model to find solution of combinatorial optimization problem in 1992. Since then, research in the development of ant based algorithms has get momentum, resulting in a large number of algorithms and applications [10]. The principle of these methods is based on the way ants search for food and find their way back to the nest. During trips of ants a chemical trail called pheromone is left on the ground. The role of pheromone is to guide the other ants towards the target point. For one ant, the path is chosen according to the quantity of pheromone. Presently successful ACO algorithms are available [1, 2, 4, 25, 33] to solve different combinatorial optimization problems. Here, to solve our problems (both crisp and fuzzy) basic ACO [8] algorithm is little modified and is presented below. In the algorithm τ_{ij} is the amount of pheromone left on the path joining the nodes i and j. For STSP τ_{ijv} is the amount of pheromone left on the path of conveyance v joining the nodes i and j. Path[k] represents the path of k-th ant. Path[k][1] represents starting node, Path[k][2] represents second node to be visited and so on. V[k] represents the conveyance used from different nodes by k-th ant. V[k][i] represents conveyance used by k-th ant to travel from node Path[k][i] to Path[k][$i+1$], for $i = 1, 2, ...$ $N-1$, and V[k][N] to travel from node Path[k][N] to Path[k][1].

Begin
Set iteration counter $t = 0$ and maximum iteration number M_0.
Place n ants at node 1.
For $i = 1, 2, ...,N$ do
  For $j = 1, 2,...,N$ do
    Initialize τ_{ij} [τ_{ijv} $\forall v$ for STSP]
  End For.
End For.
While ($t \leq M_0$) do
  For each ant $k = 1, 2, ..., n$ do //construct path of k-th ant
   DO
    Set Path[k][1] = 1
    Set NODE=$\{2, ..., N\}$
    Repeat
      Let the present position of the ant be node i and l nodes has already been visited including the starting node, i.e., Path[k][l] $= i$
      For TSP select next node j from NODE, depending upon the pheromone τ_{ij} left on the link (i,j). For STSP next node j is selected for visit by the the traveller using conveyance v (i.e., V[k][i] $= v$) depending upon the pheromone τ_{ijv}. Roulette-Wheel selection process [24] is used for this selection in both the cases.
      Add link (i, j) to Path[k], i.e., set Path[k][$l + 1$] $= j$.
      NODE=NODE-$\{j\}$
    Until NODE=ϕ [ϕ is NULL set]
   While (Path[k] does not satisfy the environmental effect constraint)
  End For.
  For $i = 1, 2, ... ,N$ do
    For $j = 1, 2, ... ,N$ do

Evaporate τ_{ij} $[\tau_{ijv}, \forall v$ for STSP]
End For.
End For.
For each ant $k = 1, 2, \ldots, n$ do
For $l = 1, 2, \ldots\ N - 1$ do
$i =$PATH$[k][l]$
$j =$PATH$[k][l+1]$
Update τ_{ij} $[\tau_{ijV[k][l]}$ for STSP]
End For.
$i =$PATH$[k][N]$
$j =$PATH$[k][1]$
Update τ_{ij} $[\tau_{ijV[k][N]}$ for STSP]
End For.
$t = t + 1$
End While
Output: path with minimum cost
End

4.1. ACO Procedures for the Proposed TSPs.

4.1.1. **Representation.** An integer variable n is used to represent number of ants in the system. A two dimensional integer array PATH$[n][N]$ is used to represent paths of different ants. PATH$[k]$ represents path of k-th ant. Path$[k][1]$ represents starting node, Path$[k][2]$ represents second node to be visited and so on and PATH$[k][N]$ is the last node visited by the ant-k. For STSP another two dimensional integer array V$[n][N]$ is used to represent different conveyances used by different ants to travel from different nodes. V$[k][i]$ represents conveyance used by k-th ant to travel from i-th node (PATH$[k][i]$) to $(i+1)$-th node (PATH$[k][i+1]$).

4.1.2. **Pheromone Initialization.** As aim of a TSP is to minimize the cost, it is assumed that initial value of $\tau_{ij} = 1/C(i,j)$. Similarly for STSP it is assumed that $\tau_{ijk} = 1/C(i,j,k)$.

4.1.3. **Path Selection.** Let recent position of the ant-$k$be node-i and it is the l-th node visited by the ant, i.e., PATH$[k][l] = i$. Then next node $j \in$NODE is selected by the ant with a probability p_{ij} given by the formula [10]

$$p_{ij} = \frac{T_{ij}^{\alpha}}{\sum\limits_{j \in NODE} T_{ij}^{\alpha}}$$

where α is a positive constant used to amplify the influence of pheromone concentrations and NODE is the set of nodes not visited by the ant. For selection of path using above probability Roulette-Wheel selection process [24] is used. Once a node is selected it is discarded from NODE.

For STSP next node j ∈NODE is selected for visiting by the traveller using conveyances v (i.e., V[k][i] $= v$) with a probability p_{ijv} given by the formula

$$p_{ijv} = \frac{T_{ijv}^{\alpha}}{\sum_{j \in NODE} \sum_{v=1}^{M} T_{ijv}^{\alpha}}$$

where α is a positive constant as described above. In this case Roulette-Wheel selection process is also used. Here, to solve the problem α is taken 1.6.

4.1.4. **Pheromone Evaporation.** For evaporation of pheromone the following formulas are used

$$\tau_{ij} = (1-\rho)\tau_{ij}, \;\; \tau_{ijk} = (1-\rho)\tau_{ijk}$$

where $\rho \in [0, 1]$. The constant ρ, specifies the rate at which pheromone evaporates, causing ants to forget previous decisions. Here, to solve the problem ρ is taken 0.2.

4.1.5. **Pheromone Updating.** Once all ants have constructed their complete tour pheromone is increased on the paths through which the ants move. If COST(k) be the cost of PATH[k], then for this path $\tau_{PATH[k][i]PATH[k][i+1]}$ is increased by 1/COST(k). As our aim is to minimize the cost this formula is used. Similarly for STSP if COST(k) be the cost of (PATH[k], V[k]), $\tau_{PATH[k][i]PATH[k][i+1]V[k][i]}$ is increased by 1/COST(k).

4.1.6. **Implementation.** With the above function and values the algorithm is implemented using C-programming language.

5. **Genetic Algorithm**

Genetic Algorithms are exhaustive search algorithms based on the mechanics of natural selection and genesis (crossover, mutation etc.) and have been developed by Holland, his colleagues and students at the University of Michigan [13]. Because of its generality and other advantages over conventional optimization methods it has been successfully applied to different decision making problems [24].

In natural genesis, we know that chromosomes are the main carriers of hereditary information from parent to offspring and that genes, which present hereditary factors, are lined up in chromosomes. At the time of reproduction, crossover and mutation take place among the chromosomes of parents. In this way hereditary factors of parents are mixed-up and carried over to their offsprings. Again Darwinian principle states that only the fittest animals can survive in nature. So a pair of fittest parents normally reproduces a better offspring.

The above-mentioned phenomenon is followed to create a genetic algorithm for an optimization problem. Here potential solutions of the problem are analogous with the chromosomes and chromosome of better offspring with the better solution of the problem. Crossover and mutation happen among a set of potential solutions to get a new set of solutions and it continues until terminating conditions are encountered. Here a GA is used to solve our TSP in both the crisp and fuzzy environment. The proposed GA and its procedures are presented below.

GA Algorithm

1. Begin
2. Initialize max generation number M_0, population size(pop_size), probability of crossover(p_c) & mutation (p_m).
3. Set iteration counter $t = 0$.
2. Randomly generate initial population $p(t)$
3. Evaluate initial population $p(t)$
4. While $t \leq M_0$ do
   a. $t \leftarrow t + 1$.
   b. Select $p(t)$ from $p(t-1)$.
   c. Alter (crossover and mutate) $p(t)$.
   d. Evaluate $p(t)$.
7. End While
8. print optimum result
9. End

5.1. GA procedures for the proposed TSPs.

5.1.1. **Representation.** Here a complete tour on N cities represents a solution. So a 'N dimensional integer vector' $X_i = (x_{i1}, x_{i2}, ..., x_{iN})$ is used to represent a solution, where x_{i1}, x_{i2}, ... , x_{iN} represent N consecutive cities in a tour. For solid TSP another integer vector $V_i = (v_{i1}, v_{i2}, ..., v_{iN})$ is used to represent the conveyances types used travel between different cities. Here v_{ij} represents the conveyance (an integer) used to travel from city x_{ij} to $x_{i(j+1)}$ for j=1, 2, ..., N-1 and v_{iN} represents the conveyance type used to travel from city x_{iN} to x_{i1}.

5.1.2. **Initialization.** pop_size number of such solutions $X_i = (x_{i1}, x_{i2}, ...x_{iN})$, $i = 1, 2, ...,$ pop_size, are randomly generated by random number generator, such that each solution satisfies the constraints of the problem. A separate sub_function check_constrint(X_i) is used for this purpose. For STSP another integer vector $V_i = (v_{i1}, v_{i2}, ..., v_{iN})$ is randomly generated corresponding to the solution X_i, to represent the conveyances types used to travel between different cities. So in that case (X_i, V_i) represent a solution.

5.1.3. **Evaluation Process.** To find fitness of a solution X_i [(X_i, V_i) for STSP], following two steps are used-

- Calculate objective function value OBJ_i for the solution X_i [(X_i, V_i) for STSP].
- As the problems are minimization type take MVAL-OBJ_i as fitness, FIT_i, of X_i [(X_i, V_i) for STSP], where MVAL is a sufficiently large value to make the fitness positive.

(d)Roulette-wheel selection process for mating pool: The following steps are followed for selection of $p(t)$ from $p(t-1)$ [24]:

(i) Find total fitness of the population, $p(t-1)$, $F_{tot} = \sum_{j=1}^{pop\_size} FIT_j$

(ii) Calculate the probability of selection p_i of each solution X_i [(X_i, V_i) for STSP] by the formula $p_i = \frac{FIT_i}{F_{tot}}$.

(iii) Calculate the cumulative probability q_i for each solution X_i [(X_i, V_i) for STSP] by the formula $q_i = \sum_{k=1}^{i} p_k$

(iv) Generate a random number r from the range [0,1].

(v) If $r < q_1$ then select X_1 [(X_1, V_1) for STSP] otherwise select $X_j (2 \le j \le n)$ [(X_j, V_j) for STSP] where $q_{j-1} \le r < q_j$.

(vi) Repeat step (iv) and (v) pop_size times to select pop_size solutions for mating pool. Clearly one solution may be selected more than once.

(vii) Selected solution set is denoted by $p(t)$ in the proposed GA algorithm.

5.1.4. **Crossover.**

(i) Selection for crossover: For each solution of $p(t)$ generate a random number r from the range [0,1]. If $r < p_c$ then the solution is taken for crossover.

(ii) Crossover process: For simple TSP cyclic crossover process [29] is used. The cyclic crossover focuses on subsets of cities that occupy the same subset of positions in both parents. Then, these cities are copied from the first parent to the offspring (at the same positions), and the remaining positions are filled with the cities of the second parent. In this way, the position of each city is inherited from one of the two parents. However, many edges can be broken in the process, because the initial subset of cities is not necessarily located at consecutive positions in the parent tours. To illustrate the process let us consider a TSP consisting of 9 cities and consider two parents PR_1, PR_2 as below:

$$PR_1 : 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9$$

$$PR_2 : 3\ \ 4\ \ 5\ \ 1\ \ 2\ \ 9\ \ 8\ \ 7\ \ 6$$

Let CH_1, CH_2 be two children born after crossover. The mechanism of birth of CH_1, CH_2 using cycle crossover is explained by the following steps:

- Randomly generate an integer in the range [1 ... 9]. Let it be 3.
- As $PR_1[3] = 3$, 3-rd element of CH_1 is 3, i.e., $CH_1[3] = 3$.
- PR_2, is then searched to check the presence of element 3 and it has been found in the 1-st position. Then 1-st element of CH_1 is selected from the 1-st element of PR_1, i.e., $CH_1[1] = PR_1[1] = 1$.
- PR_2, is again searched for the presence of element 1 and it has occurred at the 4-th position. Thus 4-th element of PR_1 has been copied as the 4-st element of CH_1, i.e., $CH_1[4] = PR_1[4] = 4$. Similarly, following are obtained:

$$CH_1[2] = PR_1[2] = 2, \quad CH_1[5] = PR_1[5] = 5$$

This completes one cycle because 5 is seen to be present at the 3-rd position of PR_2 and the corresponding 3-rd position element of PR_1 is 3, which has already been selected as the starting element of the cycle.

– The remaining elements of CH_1 are selected directly from PR_2 as follows:

$$CH_1[6] = PR_2[6] = 9, \quad CH_1[7] = PR_1[7] = 8$$

$$CH_1[8] = PR_2[8] = 7, \quad CH_1[9] = PR_1[9] = 6$$

– Final forms of CH_1 and CH_2 are as below:

$$CH_1 : 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 9 \;\; 8 \;\; 7 \;\; 6$$

$$CH_2 : 3 \;\; 4 \;\; 5 \;\; 1 \;\; 2 \;\; 6 \;\; 7 \;\; 8 \;\; 9$$

If CH_1 satisfies the constraint of the problem then PR_1 is replaced by CH_1. Similarly if CH_2 satisfies the constraint of the problem then PR_2 is replaced by CH_2.

For STSP to made crossover on two parents (PR_1, V_1), (PR_2, V_2), the same procedure is followed by PR_1 and PR_2 to obtain CH_1 and CH_2. To keep randomness in selection of conveyances conveyance sets V_1 and V_2 remains unchanged, i.e., resultant child after crossover becomes (CH_1, V_1), (CH_2, V_2). If (CH_1, V_1) satisfies the constraint of the problem then (PR_1, V_1) is replaced by (CH_1, V_1). Similarly if (CH_2, V_2) satisfies the constraint of the problem then (PR_2, V_2) is replaced by (CH_2, V_2).

5.1.5. **Mutation.**

(i) Selection for mutation: For each solution of $p(t)$ generate a random number r from the range [0,1]. If $r < p_m$ then the solution is taken for mutation.

(ii) Mutation process: To mutate a solution $X = (x_1, x_2, ...x_N)$ of TSP select two random integers i, j in the range $[1, N]$. Then interchange x_i, x_j to get child solution. New solution, if satisfies the constraint of the problem, replaces the parent solution.

For STSP to mutate a solution (X,V), where $X = (x_1, x_2, ...x_N)$, $V = (v_1, v_2, ...v_N)$ at first an integer is randomly selected in the range [1,2]. If 1 is selected then another two random integers i, j are selected in the range $[1, N]$. Then interchange x_i, x_j to get child solution. If 2 is selected then another two random integers i and j are selected in the range $[1, N]$ and $[1, M]$ respectively. Value of v_i is replaced by j to get child solution. If child solution satisfies the constraint of the problem then it replaces the parent solution.

5.1.6. **Implementation.** With the above function and values the algorithm is implemented using C-programming language.

6. Numerical Illustration

6.1. **Classical TSP with Environmental Effect Constraints.** This problem is illustrated for ten cities (N=10). Cost and environmental effect matrices are presented in Tables- 1 and 2 respectively. Optimal paths are obtained using both ACO and GA process without and with different minimum environmental effects s_{min}

i/j	1	2	3	4	5	6	7	8	9	10
1	∞	25	28	32	20	6	35	37	40	30
2	37	∞	20	28	35	40	30	42	28	4
3	42	28	∞	30	25	35	9	32	40	30
4	28	30	7	∞	20	25	30	35	22	37
5	37	22	35	30	∞	20	25	30	9	28
6	25	30	25	8	28	∞	32	40	32	30
7	28	25	30	22	37	40	∞	10	32	20
8	20	5	32	40	35	25	40	∞	22	37
9	30	40	35	25	20	22	37	32	∞	28
10	28	30	28	20	11	32	37	40	30	∞

TABLE 1. Cost Matrix $c(i,j)$ for Ten Cities ($N = 10$)

i/j	1	2	3	4	5	6	7	8	9	10
1	-	.30	.40	.50	.23	.51	.70	.60	.56	.55
2	.51	-	.60	.56	.55	.33	.44	.45	.52	.30
3	.56	.55	-	.42	.33	.36	.50	.61	.55	.70
4	.51	.70	.60	-	.55	.33	.56	.55	.41	.42
5	.20	.30	.40	.50	-	.41	.42	.33	.36	.50
6	.51	.70	.60	.56	.55	-	.44	.45	.52	.30
7	.56	.55	.41	.42	.33	.36	-	.61	.55	.70
8	.51	.70	.60	.56	.55	.33	.56	-	.41	.42
9	.30	.36	.39	.42	.51	.62	.29	.63	-	.45
10	.42	.51	.62	.29	.63	.71	.38	.28	.48	-

TABLE 2. Environmental Effect Matrix $s(i,j)$ for Ten Cities ($N = 10$)

Method	x[1],x[2],...,x[9],x[10]	cost
ACO/GA	1,6,4,3,7,8,2,10,5,9	99

TABLE 3. Optimal Solution for Classical TSP without Environment Effect Constraints

s_{min}	Method	x[1],x[2],...,x[9],x[10]	cost
5.10	ACO/GA	3,7,2,10,5,9,8,9,6,4	131
5.20	ACO/GA	8,2,10,5,9,6,1,4,3,7	134
5.40	ACO/GA	6,4,3,7,1,8,2,10,5,9	140
5.50	ACO/GA	9,7,2,1,6,4,3,7,10,5	144

TABLE 4. Optimal Solution for Classical TSP with Environment Effect Constraints

i/j	1	2	3	4	5
1	∞	25,25.5,26	28,28.5,29	32,32.5,33	20,20.5,21
2	37,37.5,38	∞	20,20.5,21	28,28.5,29	35,35.5,36
3	42,42.5,43	28,28.5,29	∞	30,30.5,31	25,25.5,26
4	28,28.5,29	30,30.5,31	7,7.5,8	∞	20,20.5,21
5	37,37.5,38	22,22.5,23	35,35.5,36	30,30.5,31	∞
6	25,25.5,26	30,30.5,31	25,25.5,26	8,8.5,9	28,28.5,29
7	28,28.5,29	25,25.5,26	30,30.5,31	22,22.5,23	37,37.5,38
8	20,20.5,21	5,5.5,6	32,32.5,33	40,40.5,41	35,35.5,36
9	30,30.5,31	40,40.5,41	35,35.5,36	25,25.5,26	20,20.5,21
10	28,28.5,29	30,30.5,31	28,28.5,29	20,20.5,21	11,11.5,12
i/j	6	7	8	9	10
1	6,6.5,6.9	35,35.5,36	37,37.5,38	40,40.5,41	30,30.5,31
2	40,40.5,41	30,30.5,31	42,42.5,43	28,28.5,29	4,4.5,5
3	35,35.5,36	9,9.5,10	32,32.5,33	40,40.5,41	30,30.5,31
4	25,25.5,26	30,30.5,31	35,35.5,36	22,22.5,23	37,37.5,38
5	20,20.5,21	25,25.5,26	30,30.5,31	9,9.5,10	28,28.5,29
6	∞	32,32.5,33	40,40.5,41	32,32.5,33	30,30.5,31
7	40,40.5,41	∞	10,10.5,11	32,32.5,33	20,20.5,21
8	25,25.5,26	40,40.5,41	∞	22,22.5,23	37,37.5,38
9	22,22.5,23	37,37.5,38	32,32.5,33	∞	28,28.5,29
10	32,32.5,33	37,37.5,38	40,40.5,41	30,30.5,31	∞

TABLE 5. Fuzzy Cost Matrix $\tilde{c}$(i,j) for Ten Cities

and presented in Tables-3 and 4 respectively. It is observed that results obtained via both the processes are the same for e cases without and with environment effect. It is also observed that cost of the tour increases with increment of S_{min}, which agrees with reality.

6.2. **Classical Fuzzy TSP with Environmental Effect Constraints.** This problem is also illustrated for ten cities ($N = 10$). Cost and environmental effect matrices are presented in Tables-5 and 6 respectively. Other assumed parametric values are $\widetilde{S_{min}} = (5.15, 5.60, 6.0)$, $\alpha_1 = 0.9$, $\alpha_2 = 0.1$.

Optimal paths for both the optimistic DM (ODM) and pessimistic DM (PDM), i.e., solutions of problems (13) and (14) respectively are obtained using both ACO and GA processes due to different values of β_1 and β_2 and are presented in Table-8. In this case it is also observed that results obtained via both processes are almost the same. It is also observed that total cost increases by increasing β_1, β_2 in both pessimistic and optimistic cases. All these are our expectations, rigid constraint increases the total cost. The results without taking any constraint into account are also obtained and presented in Table-7. In this table objective values (OBJV) i.e. $C_1+\alpha_1*(C_2 - C_1)$ are also given, which are different in pessimistic and optimistic

i/j	1	2	3	4	5
1	-	.3,.35,.38	.4,.45,5	.5,.55,.6	.23,.25,.29
2	.51,.55,.58	-	.6,.65,.69	.8,.85,.88	.55,.58,.6
3	.56,.6,.65	.55,.58,.6	-	.42,.44,.45	.33,.35,.38
4	.51,.55,.58	.7,.75,.77	.6,.7,.73	-	.55,.58,.61
5	.51,.52,.53	.7,.71,.72	.6,.62,.63	.56,.57,.58	-
6	.20,.25,.28	.30,.35,.38	.40,.45,50	.50,.55,.60	.23,.25,.29
7	.51,.55,.58	.7,.75,.78	.6,.65,.69	.8,.85,.88,	.55,.58,.6
8	.56,.6,.65	.55,.58,.6	.41,.45,.5	.42,.44,.45	.33,.35,.38
9	.51,.55,.58	.7,.75,.77	.6,.7,.73	.56,.57,.58	.55,.58,.61
10	.51,.52,.53	.7,.71,.72	.6,.62,.63	.56,.57,.58	.55,.56,.57

i/j	6	7	8	9	10
1	.51,.6,.62	.7,.75,.78	.6,.65,.7	.56,.59,.6	.55,.56,.57
2	.33,.36,.38	.44,.48,.5	.45,.48,.51	.52,.55,.58	.3,.35,.4
3	.36,.38,.4	50,.55,.58	.61,.65,.7	.55,.58,.58	.7,.75,.8
4	.33,.36,.38	.56,.58,.62	.55,.56,.57	.41,.45,.48	.42,.45,.48
5	.33,.34,.35	.56,.57,.58	.55,.56,.57	.41,.42,.43	.42,.43,.44
6	-	.7,.75,.78	.6,.65,.7	.56,.59,.6	.55,.56,.57
7	.33,.36,.38	-	.45,.48,.51	.52,.55,.58	.3,.35,.4
8	.36,.38,.4	50,.55,.58	-	.55,.58,.58	.7,.75,.8
9	.33,.36,.38	.56,.58,.62	.55,.56,.57	-	.42,.45,.48
10	.33,.34,.35	.56,.57,.58	.55,.56,.57	.41,.42,.43	-

TABLE 6. Fuzzy Environmental Effect Matrix $\tilde{s}$(i,j) for Ten Cities

Method	DM	x[1],x[2],...,x[9],x[10]	$\widetilde{cost}$	OBJV
ACO/	PDM	4,3,7,8,2,10,5,9,1,6	99.0,104.0,108.0	107.59
GA	ODM	7,8,2,10,5,9,4,3,1,6	99.0,104.0,108.0	103.50

TABLE 7. Optimal Solution for Classical Fuzzy TSP without Environment Effect Constraints

models. For the present parametric values it is observed that the optimum cost of both models are the same as the following approaches ACO and GA.

6.3. **Solid TSP with Crisp Cost and Environmental Effect Constraints.** This problem is illustrated for ten cities (N=10) and three types of conveyances(M=3). Cost matrices are presented in Table-9, 10, 11 respectively. Similarly environmental effect matrices for different conveyances are given in Table-12, 13, 14 respectively.

Optimal paths along with the selected conveyances are obtained due to different S_{min} using both ACO and GA processes and presented in Table-16. It is observed that results obtained via both processes are almost the same. In this table, as

β_1/β_2	Method	DM	x[1],x[2],...,x[9],x[10]	$\widetilde{cost}$
.2	ACO	PDM	6,7,4,3,8,2,10,5,9,1	158.0,164.0,167.3
		ODM	1,6,4,3,7,8,2,10,5,9	99.0,104.0,108.9
	GA	PDM	6,7,4,3,8,2,10,5,9,1	158.0,164.0,167.3
		ODM	1,6,4,3,7,8,2,10,5,9	99.0,104.0,108.9
.6	ACO	PDM	1,6,10,5,9,2,4,3,7,8	170.0,175.00,179.0
		ODM	9,8,1,6,4,3,7,2,10,5	131.0,136.0,140.9
	GA	PDM	1,6,7,8,2,4,3,10,5,9	168.0,173.00,177.9
		ODM	9,8,1,6,4,3,7,2,10,5	131.0,136.0,140.9
.8	ACO	PDM	1,6,7,4,3,10,5,9,8,2	191.0,196.00,200.8
		ODM	9,8,1,6,4,3,7,2,10,5	131.0,136.0,140.9
	GA	PDM	1,6,7,4,3,10,5,9,8,2	191.0,196.00,200.8
		ODM	9,8,1,6,4,3,7,2,10,5	131.0,136.0,140.9

Table 8. Optimal Solution for Classical Fuzzy TSP with Environment Effect Constraints

i/j	1	2	3	4	5	6	7	8	9	10
1	∞	25	28	32	20	6	35	37	40	30
2	37	∞	20	28	35	40	30	42	28	4
3	42	28	∞	30	25	35	9	32	40	30
4	28	30	7	∞	20	25	30	35	22	37
5	37	22	35	30	∞	20	25	30	9	28
6	25	30	25	8	28	∞	32	40	32	30
7	28	25	30	22	37	40	∞	10	32	20
8	20	5	32	40	35	25	40	∞	22	37
9	30	40	35	25	20	22	37	32	∞	28
10	28	30	28	20	11	32	37	40	30	∞

Table 9. Cost Matrix c(i,j,1) for Ten Cities Via First Conveyances

i/j	1	2	3	4	5	6	7	8	9	10
1	∞	25.5	28.5	32.5	20.5	6.5	35.5	37.5	40.5	30.5
2	37.5	∞	20.5	28.5	35.5	40.5	30.5	42.5	28.5	4.5
3	42.5	28.5	∞	30.5	25.5	35.5	9.5	32.5	40.5	30.5
4	28.5	30.5	7.5	∞	20.5	25.5	30.5	35.5	22.5	37.5
5	37.5	22.5	35.5	30.5	∞	20.5	25.5	30.5	9.5	28.5
6	25.5	30.5	25.5	8.5	28.5	∞	32.5	40.5	32.5	30.5
7	28.5	25.5	30.5	22.5	37.5	40.5	∞	10.5	32.5	20.5
8	20.5	5.5	32.5	40.5	35.5	25.5	40.5	∞	22.5	37.5
9	30.5	40.5	35.5	25.5	20.5	22.5	37.5	32.55	∞	28.5
10	28.5	30.5	28.5	20.5	11.5	32.5	37.5	40.5	30.5	∞

Table 10. Cost Matrix c(i,j,2) for Ten Cities Via Second Conveyances

i/j	1	2	3	4	5	6	7	8	9	10
1	∞	26	29	33	21	6.9	36	38	41	31
2	37	∞	20	28	35	40	30	42	28	4
3	42	28	∞	30	25	35	9	32	40	30
4	28	30	7	∞	20	25	30	35	22	37
5	37	22	35	30	∞	20	25	30	9	28
6	25	30	25	8	28	∞	32	40	32	30
7	28	25	30	22	37	40	∞	10	32	20
8	20	5	32	40	35	25	40	∞	22	37
9	30	40	35	25	20	22	37	32	∞	28
10	28	30	28	20	11	32	37	40	30	∞

TABLE 11. Cost Matrix c(i,j,3) for Ten Cities Via Third Conveyances

i/j	1	2	3	4	5	6	7	8	9	10
1	∞	.3	.4	.5	.23	.51	.7	.6	.56	.55
2	.51	∞	.6	.8	.55	.33	.44	.45	.52	.3
3	.56	.55	∞	.42	.33	.36	.50	.61	.55	.7
4	.51	.7	.6	∞	.55	.33	.56	.55	.41	.42
5	.51	.7	.6	.56	∞	.33	.56	.55	.41	.42
6	.2	.3	.4	.5	.23	∞	.7	.6	.56	.55
7	.51	.7	.6	.8	.55	.33	∞	.45	.52	.3
8	.56	.55	.41	.42	.33	.36	.50	∞	.55	.7
9	.51	.7	.6	.56	.55	.33	.56	.55	∞	.42
10	.51	.7	.6	.8	.55	.33	.56	.55	.41	∞

TABLE 12. Environmental Effect Matrix s(i,j,1) for Ten Cities Via First Conveyances

i/j	1	2	3	4	5	6	7	8	9	10
1	∞	.35	.45	.55	.25	.6	.75	.65	.59	.56
2	.55	∞	.65	.85	.58	.36	.48	.48	.55	.35
3	.6	.58	∞	.44	.35	.38	.55	.65	.58	.75
4	.55	.75	.7	∞	.58	.36	.58	.56	.45	.45
5	.52	.71	.62	.57	∞	.34	.57	.56	.42	.43
6	.25	.35	.45	.55	.25	∞	.75	.65	.59	.56
7	.55	.75	.65	.85	.58	.36	∞	.48	.55	.35
8	.6	.58	.45	.44	.35	.38	.55	∞	.58	.75
9	.55	.75	.7	.57	.58	.36	.58	.56	∞	.45
10	.52	.71	.62	.57	.56	.34	.57	.56	.42	∞

TABLE 13. Environmental Effect Matrix s(i,j,2) for Ten Cities Via Second Conveyances

i/j	1	2	3	4	5	6	7	8	9	10
1	∞	.38	.5	.6	.29	.62	.78	.7	.6	.57
2	.58	∞	.69	.88	.6	.38	.5	..51	.58	.5
3	.65	.6	∞	.45	.38	.4	.58	.7	.58	.8
4	.58	.77	.73	∞	.61	.38	.62	.57	.48	.48
5	.53	.72	.63	.58	∞	.35	.58	.57	.43	.44
6	.28	.38	.5	.6	.29	∞	.78	.7	.6	.57
7	.58	.78	.69	.88	.6	.38	∞	.51	.58	.4
8	.65	.6	.5	.45	.38	.4	.58	∞	.58	.8
9	.58	.77	.73	.58	.61	.38	.62	.57	∞	.48
10	.53	.72	.63	.58	.57	.35	.58	.57	.43	∞

TABLE 14. Environmental Effect Matrix s(i,j,3) for Ten Cities Via Third Conveyances

Method	(x[1],v[1]),(x[2],v[2]),...,(x[10],v[10])	*cost*
ACO	1,1;6,1;4,1;3,1;7,3;8,1;2,1;10,2;5,2;9,3	102.5
GA	5,1;4,3;3,2;7,1;8,1;2,3;10,2;5,1;9,2;1,3	104.0

TABLE 15. Optimal Solution for Solid TSP without Environment Effect Constraints

S_{min}	Method	x[1],v[1]; x[2],v[2];...;x[10],v[10]	*cost*
5.6	ACO	1,3; 6,3; 4,2; 3,2; 7,2; 2,3; 10,1; 5,1; 9,3; 8,2	136.3
	GA	5,1; 9,1; 8,3; 1,2; 6,1; 4,3; 3,1; 7,2; 2,3; 10,1	135.6
5.7	ACO	1,3; 6,3; 4,3; 3,3; 7,2; 2,3; 10,1; 5,1; 9,3; 8,3	138.3
	GA	1,1; 6,2; 4,3; 3,2; 7,3; 2,3; 10,3; 5,1; 9,1; 8,3	137.5
5.8	ACO	1,1; 6,3; 7,1; 4,3; 3,2; 8,2; 2,1; 10,3; 5,3; 9,1	163.3
	GA	1,1; 6,2; 7,1; 4,2; 3,2; 8,3; 2,1; 10,3; 5,3; 9,1	162.7

TABLE 16. Optimal Solution for Solid TSP with Crisp Cost and Environment Effect Constraints

expected, the results of two processes increase with the increase of S_{min}. The results without taking constrains are also obtained and presented in Table-15.

6.4. **Solid TSP with Fuzzy Costs and Environmental Effect Constraints.** This problem is also illustrated for ten cities (N=10) and three types of conveyances(M=3). Cost matrix for M=1, M=2 and M=3 are presented in Table-17, 18 and 19 respectively. Similarly environmental effect matrix for different conveyances are given in Table-20, 21 and 22 respectively. Other assumed parametric values are $\tilde{S_{min}} = (5.6, 5.7, 5.8)$, $\alpha_3 = 0.9$, $\alpha_4 = 0.1$. Optimal paths along with the selected conveyances are obtained using both ACO and GA process for both the optimistic and pessimistic DMs due to different values of β_3, β_4 and are presented in Table-24. It is observed that in few cases ACO gives slightly better results than

i/j	1	2	3	4	5
1	-	25,25.2,25.4	28,28.2,28.5	32,32.2,32.5	20,20.2,20.5
2	37,37.2,37.5	-	20,20.2,20.5	28,28.2,28.5	35,35.2,35.5
3	42,42.2,42.5	28,28.2,28.5	-	30,30.2,30.5	25,25.2,25.4
4	28,28.2,28.5	30,30.2,30.5	7,7.2,7.5	-	20,20.2,20.5
5	37,37.2,37.5	22,22.2,22.	35,35.2,35.5	30,30.2,30.5	-
6	25,25.2,25.4	30,30.2,30.5	25,25.2,25.4	8,8.2,8.4	28,28.2,28.5
7	28,28.2,28.5	25,25.2,25.4	30,30.2,30.5	22,22.2,22.5	37,37.2,37.5
8	20,20.2,20.5	5,5.2,5.5	32,32.2,32.5	40,40.2,40.5	35,35.2,35.5
9	30,30.2,30.5	40,40.2,40.5	35,35.2,35.5	25,25.2,25.4	20,20.2,20.5
10	28,28.2,28.5	30,30.2,30.5	28,28.2,28.5	20,20.2,20.5	11,11.2,11.4
i/j	6	7	8	9	10
1	6,6.2,6.5	35,35.2,35.5	37,37.2,37.5	40,40.2,40.5	30,30.2,30.5
2	40,40.2,40.5	30,30.2,30.5	42,42.2,42.5	28,28.2,28.5	4,4.2,4.5
3	35,35.2,35.5	9,9.2,9.5	32,32.2,32.5	40,40.2,40.5	30,30.2,30.5
4	25,25.2,25.4	30,30.2,30.5	35,35.2,35.5	22,22.2,22.5	37,37.2,37.5
5	20,20.2,20.5	25,25.2,25.4	30,30.2,30.5	9,9.2,9.5	28,28.2,28.5
6	-	32,32.2,32.5	40,40.2,40.5	32,32.2,32.5	30,30.2,30.5
7	40,40.2,40.5	-	10,10.2,10.5	32,32.2,32.5	20,20.2,20.5
8	25,25.2,25.4	40,40.2,40.5	-	22,22.2,22.5	37,37.1,37.4
9	22,22.2,22.5	37,37.1,37.4	32,32.2,32.5	-	28,28.2,28.5
10	32,32.2,32.5	37,37.1,37.4	40,40.2,40.5	30,30.2,30.5	-

TABLE 17. Fuzzy Cost Matrix $\tilde{c}(i, j, 1)$ for Ten Cities Via First Conveyances

i/j	1	2	3	4	5
1	-	25.5,25.6,25.8	28.5,28.6,28.8	32.5,32.6,32.8	20.5,20.6,20.8
2	37.5,37.6,37.8	-	20.5,20.6,20.8	28.5,28.6,28.8	35.5,35.6,35.8
3	42.5,42.6,42.8	28.5,28.6,28.8	-	30.5,30.2,30.5	25.5,25.6,25.8
4	28.5,28.6,28.8	30.5,30.2,30.5	7.5,7.6,7.8	-	20.5,20.6,20.8
5	37.5,37.6,37.8	22.5,22.6,22.8	35.5,35.6,35.	30.5,30.2,30.5	-
6	25.5,25.6,25.8	30.5,30.2,30.5	25.5,25.6,25.8	8.5,8.6,8.8	28.5,28.6,28.8
7	28.5,28.6,28.8	25.5,25.6,25.8	30.5,30.2,30.5	22.5,22.6,22.8	37.5,37.6,37.8
8	20.5,20.6,20.8	5.5,5.6,5.8	32.5,32.6,32.8	40.5,40.6,40.8	35.5,35.6,35.8
9	30.5,30.2,30.5	40.5,40.6,40.8	35.5,35.6,35.8	25.5,25.6,25.8	20.5,20.6,20.8
10	28.5,28.6,28.8	30.5,30.2,30.5	28.5,28.6,28.8	20.5,20.6,20.8	11.5,11.6,11.8
i/j	6	7	8	9	10
1	6.5,6.6,6.8	35.5,35.6,35.8	37.5,37.6,37.8	40.5,40.6,40.8	30.5,30.2,30.5
2	40.5,40.6,40.8	30.5,30.2,30.5	42.5,42.6,42.8	28.5,28.6,28.8	4.5,4.6,4.8
3	35.5,35.6,35.8	9.5,9.6,9.8	32.5,32.6,32.8	40.5,40.6,40.8	30.5,30.2,30.5
4	25.5,25.6,25.8	30.5,30.2,30.5	35.5,35.6,35.8	22.5,22.6,22.8	37.5,37.6,37.8
5	20.5,20.6,20.8	25.5,25.6,25.8	30.5,30.2,30.5	9.5,9.6,9.8	28.5,28.6,28.8
6	-	32.5,32.6,32.8	40.5,40.6,40.8	32.5,32.6,32.8	30.5,30.2,30.5
7	40.5,40.6,40.8	-	10.5,10.6,10.8	32.5,32.6,32.8	20.5,20.6,20.8
8	25.5,25.6,25.8	40.5,40.6,40.8	-	22.5,22.6,22.8	37.5,37.6,37.7
9	22.5,22.6,22.8	37.5,37.6,37.7	32.5,32.6,32.8	-	28.5,28.6,28.8
10	32.5,32.6,32.8	37.5,37.6,37.7	40.5,40.6,40.8	30.5,30.2,30.5	-

TABLE 18. Fuzzy Cost Matrix $\tilde{c}(i, j, 2)$ for Ten Cities Via Second Conveyances

i/j	1	2	3	4	5
1	-	26,26.5,26.8	29,29.3,29.8	33,33.2,3.5	21,21.3,21.6
2	38,38.2,38.5	-	21,21.3,21.6	29,29.3,29.8	36,36.2,36.6
3	43,43.2,43.5	29,29.3,29.8	-	31,31.2,31.5	26,26.5,26.8
4	29,29.3,29.8	31,31.2,31.5	8,8.2,8.5	-	21,21.3,21.6
5	38,38.2,38.	23,23.2,23.6	36,36.2,36.6	31,31.2,31.5	-
6	26,26.5,26.8	31,31.2,31.5	26,26.5,26.5	9,9.2,9.5	29,29.3,29.8
7	29,29.3,29.8	26,26.5,26.5	31,31.2,31.5	23,23.2,23.6	38,38.2,38.5
8	21,21.3,21.6	6,6.3,6.5	33,33.2,3.5	41,41.2,41.5	36,36.2,36.6
9	31,31.2,31.5	41,41.2,41.5	36,36.2,36.6	26,26.5,26.5	21,21.3,21.6
10	29,29.3,29.8	31,31.2,31.5	29,29.3,29.8	21,21.3,21.6	12,12.3,12.5

i/j	6	7	8	9	10
1	6.9,7.0,7.3	36,36.2,36.6	38,38.2,38.5	41,41.2,41.5	31,31.2,31.5
2	41,41.2,41.5	31,31.2,31.5	43,43.2,43.5	29,29.3,29.8	5,5.2,5.5
3	36,36.2,36.6	10,10.2,10.5	33,33.2,3.5	41,41.2,41.5	31,31.2,31.5
4	26,26.5,26.8	31,31.2,31.5	36,36.2,36.6	23,23.2,23.6	38,38.2,38.5
5	21,21.3,21.6	26,26.5,26.8	31,31.2,31.5	10,10.2,10.5	29,29.3,29.8
6	-	33,33.2,3.5	41,41.2,41.5	33,33.2,3.5	31,31.2,31.5
7	41,41.2,41.5	-	11,11.2,11.5	33,33.2,3.5	21,21.3,21.6
8	26,26.5,26.5	41,41.2,41.5	-	23,23.2,23.6	38,38.1,38.2
9	23,23.2,23.6	38,38.1,38.2	33,33.2,3.5	-	29,29.3,29.8
10	33,33.2,33.5	38,38.1,38.2	41,41.2,41.5	31,31.2,31.5	-

TABLE 19. Fuzzy Cost Matrix $\tilde{c}(i, j, 3)$ for Ten Cities Via Second Conveyances

i/j	1	2	3	4	5
1	∞	.3,.31,.32	.4,.41,.42	.5,.51,.52	.23,.24,.25
2	.51,.55,.58	∞	.41,.45,.5	.44,.48,.5	.51,.55,.58
3	.2,.21,.22	.3,.31,.32	∞	.5,.51,.52	.23,.24,.25
4	.51,.55,.58	.56,.6,.65	.41,.45,.5	∞	.51,.55,.58
5	.2,.21,.22	.3,.31,.32	.4,.41,.42	.5,.51,.52	∞
6	.51,.55,.58	.56,.6,.65	.41,.45,.5	.44,.48,.5	.51,.55,.58
7	.2,.21,.22	.3,.31,.32	.4,.41,.42	.5,.51,.52	.23,.24,.25
8	.51,.55,.58	.56,.6,.65	.41,.45,.5	.44,.48,.5	.51,.55,.58
9	.2,.21,.22	.3,.31,.32	.4,.41,.42	.5,.51,.52	.23,.24,.25
10	.51,.55,.58	.56,.6,.65	.41,.45,.5	.44,.48,.5	.51,.55,.58

i/j	6	7	8	9	10
1	.51,.52,.53	.7,.71,.72	.6,.61,.62	.56,.57,.58	.55,.56,.57
2	.51,.55,.58	.33,.36,.38	.36,.38,.4	.51,.55,.58	.41,.45,.5
3	.51,.52,.53	.7,.71,.72	.6,.61,.62	.56,.57,.58	.55,.56,.57
4	.51,.55,.58	.33,.36,.38	.36,.38,.4	.51,.55,.58	.41,.45,.5
5	.51,.52,.53	.7,.71,.72	.6,.61,.62	.56,.57,.58	.55,.56,.57
6	∞	.33,.36,.38	.36,.38,.4	.51,.55,.58	.41,.45,.5
7	.51,.52,.53	∞	.6,.61,.62	.56,.57,.58	.55,.56,.57
8	.51,.55,.58	.33,.36,.38	∞	.51,.55,.58	.41,.45,.5
9	.51,.52,.53	.7,.71,.72	.6,.61,.62	∞	.55,.56,.57
10	.51,.55,.58	.33,.36,.38	.36,.38,.4	.51,.55,.58	∞

TABLE 20. Fuzzy Environmental Effect Matrix $\tilde{S}(i, j, 1)$ for Ten Cities Using First Conveyances

GA and some other cases GA gives better result than ACO. As before the result increases with the increase of β_3, β_4. The results without taking any constrains are presented in Table-23.

6.5. **TSPLIB Data Set.** In this section some standard data sets are used to test the effectiveness of our algorithm for solving the proposed problem. For this purpose some standard benchmark TSPs from TSPLIB are used. These standard TSPs are classical, but our requirement is STSP. A STSP with single conveyance facility, without environmental effect constraint is the same as a classical TSP. For this reason our algorithms are tested for TSPs from TSPLIB, taking only one conveyance facility and ignoring the environmental effects. Our observed results for different problems are presented in Table-25. The same problems were solved by Wang et al. [34] using Self Adaptive Genetic Algorithm (SAGA) and SGA. Their

i/j	1	2	3	4	5
1	∞	.35,.36,.37	.45,.46,.47	.55,.56,.57	.25,.26,.27
2	.7,.75,.78	∞	.42,.44,.45	.45,.48,.51	.7,.75,.77
3	.25,.26,.27	.35,.36,.37	∞	.55,.56,.57	.25,.26,.27
4	.7,.75,.78	.55,.58,.6	.42,.44,.45	∞	.7,.75,.77
5	.25,.26,.27	.35,.36,.37	.45,.46,.47	.55,.56,.57	∞
6	.7,.75,.78	.55,.58,.6	.42,.44,.45	.45,.48,.51	.7,.75,.77
7	.25,.26,.27	.35,.36,.37	.45,.46,.47	.55,.56,.57	.25,.26,.27
8	.7,.75,.78	.55,.58,.6	.42,.44,.45	.45,.48,.51	.7,.75,.77
9	.25,.26,.27	.35,.36,.37	.45,.46,.47	.55,.56,.57	.25,.26,.27
10	.7,.75,.78	.55,.58,.6	.42,.44,.45	.45,.48,.51	.7,.75,.77

i/j	6	7	8	9	10
1	.6,.61,.62	.75,.76,.77	.65,.66,.67	.59,.6,.61	.56,.57,.58
2	.7,.75,.77	.44,.48,.5	.50,.55,.58	.6,.7,.73	.50,.55,.58
3	.6,.61,.62	.75,.76,.77	.65,.66,.67	.59,.6,.61	.56,.57,.58
4	.7,.75,.77	.44,.48,.5	.50,.55,.58	.6,.7,.73	50,.55,.58
5	.6,.61,.62	.75,.76,.77	.65,.66,.67	.59,.6,.61	.56,.57,.58
6	∞	.44,.48,.5	.50,.55,.58	.6,.7,.73	.50,.55,.58
7	.6,.61,.62	∞	.65,.66,.67	.59,.6,.61	.56,.57,.58
8	.7,.75,.77	.44,.48,.5	∞	.6,.7,.73	.50,.55,.58
9	.6,.61,.62	.75,.76,.77	.65,.66,.67	∞	.56,.57,.58
10	.7,.75,.77	.44,.48,.5	.50,.55,.58	.6,.7,.73	∞

TABLE 21. Fuzzy Environmental Effect Matrix $\tilde{S}(i,j,2)$ for Ten Cities Using Second Conveyances

i/j	1	2	3	4	5
1	∞	.38,.39,.40	.5,.51,.52	.6,.61,.62	.29,.30,.31
2	.6,.65,.69	∞	.33,.35,.38	.52,.55,.58	.6,.7,.73
3	.28,.29,.29	.38,.39,.40	∞	.6,.61,.62	.29,.30,.31
4	.6,.65,.69	.41,.45,.5	.33,.35,.38	∞	.6,.7,.73
5	.27,.28,.29	.38,.39,.40	.5,.51,.52	.6,.61,.62	∞
6	.6,.65,.69	.41,.45,.5	.33,.35,.38	.52,.55,.58	.6,.7,.73
7	.28,.29,.30	.38,.39,.40	.5,.51,.52	.6,.61,.62	.29,.30,.31
8	.6,.65,.69	.41,.45,.5	.33,.35,.38	.52,.55,.58	.6,.7,.73
9	.28,.29,.29	.38,.39,.40	.5,.51,.52	.6,.61,.62	.29,.30,.31
10	.6,.65,.69	.41,.45,.5	.33,.35,.38	.52,.55,.58	.6,.7,.73

i/j	6	7	8	9	10
1	.62,.63,.64	.78,.79,.8	.7,.71,.72	.6,.61,.62	.57,.58,.6
2	.6,.7,.73	.45,.48,.51	.61,.65,.7	.7,.75,.77	.61,.65,.7
3	.62,.63,.64	.78,.79,.8	.7,.71,.72	.6,.61,.62	.57,.58,.6
4	.6,.7,.73	.45,.48,.51	.61,.65,.7	.7,.75,.77	.61,.65,.7
5	.62,.63,.64	.78,.79,.80	.70,.71,.72	.60,.61,.62	.57,.58,.6
6	∞	.45,.48,.51	.61,.65,.7	.7,.75,.77	.61,.65,.7
7	.62,.63,.64	∞	.7,.71,.72	.6,.61,.62	.57,.58,.6
8	.6,.7,.73	.45,.48,.51	∞	.7,.75,.77	.61,.65,.7
9	.62,.63,.64	.78,.79,.8	.7,.71,.72	∞	.57,.58,.6
10	.6,.7,.73	.45,.48,.51	.61,.65,.7	.7,.75,.77	∞

TABLE 22. Fuzzy Environmental Effect Matrix $\tilde{S}(i,j,3)$ for Ten Cities Using Third Conveyances

Method	DM	x[1],v[1];x[2],v[2];...;x[10],v[10]	$\widetilde{cost}$
ACO	PDM	1,2;6,2;4,1;3,1;7,3;8,2;2,1;10,1;5,2;9,3	103.5, 105.6, 108.1
	ODM	1,1;6,1;4,1;3,1;7,3;8,3;2,1;10,3;5,2;9,3	103.5, 104.0, 107.0
GA	PDM	1,2;6,2;4,2;3,1;7,1;8,3;2,1;10,2;5,1;9,3	103.0, 104.7, 107.8
	ODM	3,3;7,2;8,2;2,2;10,1;5,1;9,3;1,1;6,2;4,1	103.0, 104.6, 107.1

TABLE 23. Optimal Solution for Solid TSP without Environment Effect Constraints

results along with the optimum results of the problems are also presented in Table-25, to check the efficiency of our algorithms. It is observed that our algorithms give better results than that of either SAGA or SGA using less number of iterations. Best tours obtained using our ACO for these problems are presented in Appendix-A.

The above maintained standard benchmark TSPs from TSPLIB are redefined for the proposed Solid TSP, and results for different problems are presented in Table-26. For this purpose three types of conveyance are used for each problem.

β_3/β_4	Method	DM	x[1],v[1];x[2],v[2];...;x[10],v[10]	$\widetilde{cost}$
.2	ACO	PDM	1,1;2,1;10,2;5,2;9,1;6,3;4,1;3,2;7,2;8,2	128.5,130.1,132.4
		ODM	1,1;6,1;4,1;3,1;7,2;8,1;2,2;10,2;5,1;9,3	101.5,103.1,105.8
	GA	PDM	1,1;2,1;10,2;5,2;9,1;6,3;4,1;3,2;7,2;8,2	128.5,130.1,132.4
		ODM	2,2;10,1;5,2;9,1;1,3;6,1;4,1;3,2;7,2;8,1	101.4,103.0,105.5
.6	ACO	PDM	1,1;6,2;5,3;9,1;4,2;3,1;7,1;8,1;2,1;10,1	133.3,134.3,136.2
		ODM	1,2;6,3;4,1;3,1;7,2;8,1;2,2;10,2;5,2;9,3	103.5,105.0,107.5
	GA	PDM	3,2;7,3;8,2;1,3;2,3;10,1;5,3;9,2;6,2;4,1	131.0,132.9,135.4
		ODM	10,2;5,1;9,1;1,2;6,3;4,2;3,3;7,1;8,1;2,2	103.0,104.6,107.25
.8	ACO	PDM	1,2;6,1;2,2;10,2;5,1;9,1;4,1;3,3;7,2;8,1	134.0,135.5,138.1
		ODM	1,2;6,3;4,1;3,1;7,3;8,1;2,2;10,2;5,2;9,3	103.64,105.2,107.8
	GA	PDM	10,1;1,2;6,2;5,1;9,3;4,1;3,1;7,1;8,1;2,2	133.5,135.5,137.9
		ODM	4,1;3,2;7,1;8,1;2,2;10,2;5,3;9,2;1,3;6,3	103.9,105.0,107.0

TABLE 24. Optimal Solution for Solid TSP with Fuzzy Cost and Environment Effect Constraints with $\widetilde{S_{min}} = (5.6, 5.7, 5.8)$

Problem Name	Size	Optimal by TSPLIB	Method	Generation	Cost of best tour
br17	17	39	ACO	500	39
			GA	500	39
			SAGA	1000	39
ftv33	34	1286	ACO	1000	1343
			GA	1000	1349
			SAGA	4000	1453
ftv55	56	1608	ACO	1000	1635
			GA	1000	1663
			SAGA	4000	1672
ry48p	48	14422	ACO	1000	14915
			GA	1000	14915
			SGA	4000	15024
ft70	70	38673	ACO	1000	39702
			GA	1000	39702
			SGA	4000	41056

TABLE 25. Results of TSPLIB without Environmental Effect for Standard Five Cases

Problem Name (Redefined)	Size	S_{min}	Method	Generation	Cost of Best Tour
Redefined br17	17	11.0	ACO	500	102.85
			GA	500	102.85
Redefined ftv33	34	23.0	ACO	1000	1340.02
			GA	1000	1343.80
Redefined ftv55	56	36.5	ACO	1000	1567.03
			GA	1000	1569.37
Redefined ry48p	48	34.5	ACO	1000	15139.04
			GA	1000	15372.70
Redefined ft70	70	50.0	ACO	1000	49283.38
			GA	1000	49346.46

TABLE 26. Results of Solid TSP Using Redefined TSPLIB Data Base with Environmental Effect for Standard Five Cases in Crisp Environment

If c_{ij} is actual cost of travel from i-th city to j-th city for a benchmark TSP, then c_{ij1}, c_{ij2} and c_{ij3} are taken as cost of corresponding Solid TSP to travel from i-th city to j-th city using first, second and third type of conveyance respectively where $c_{ij1} = c_{ij} * \xi_{ij1}$, $c_{ij2} = c_{ij}$ and $c_{ij3} = c_{ij} * \xi_{ij3}$ where ξ_{ij1} and ξ_{ij3} are two randomly generated data in the range$(1, 5)$ and $*$ is the binary operation randomly selected from the set$\{+, -\}$. Finally environmental effect due to different types of conveyances are randomly generated in the range$(.4, .9)$.

Problem Name	$\widetilde{S_{min}}$	Meth -od	Gene- ration	Cost of $\widetilde{\text{Best}}$ Tour	Obje- ctive
Redefined $br17$	8.0,9.0,9.1	ACO	500	97.33,103.00,107.72	102.43
		GA	500	97.33,103.00,107.72	102.43
Redefined ftv33	21.0,22.5,22.8	ACO	1000	1328.87,1339.54,1349.70	1338.47
		GA	1000	1330.57,1341.04,1350.30	1339.99
Redefined ftv55	36.0,36.5,36.8	ACO	1000	1648.78,1665.96,1684.88	1664.25
		GA	1000	1660.62,1677.22,1695.48	1675.56
Redefined ry48p	32.0,33.0,33.8	ACO	1000	16620.44,16635.09,16649.03	16633.63
		GA	1000	16650.96,16664.97,16679.29	16663.57
Redefined ft70	43.6,48.7,55.8	ACO	1000	43425.03,43445.75,43467.57	43443.68
		GA	1000	43425.03,43445.75,43467.57	43443.68

Table 27. Results of Solid TSP Using Redefined TSPLIB Data Base with Environmental Effect for Standard Five Cases in Fuzzy Environment

Redefined Bench Mark TSPs for Solid TSP are again redefined for Solid TSP in fuzzy environment and results are presented in Table-27. Here fuzzy cost and data are taken as TFN type. Mid value of these fuzzy data are taken as corresponding data of crisp problem. Then a random number is generated in the range $(.1, .5)$ and is subtracted from mid value to get left component of the data. Similarly another number is generated in the range $(.1, .5)$ and is added to mid value to get right component. For the generation of data of fuzzy environment effect procedure is the same as fuzzy cost, but the random data is generated in the range $(.05, .08)$.

7. Conclusion

Here for the first time, a constrained solid TSP is modelled where the salesman can choose suitable conveyance for the tour from one city to another against different environmental effects. Goal of the traveller is to visit all the cities with a minimum cost such that a maximum environmental effect(in positive sense) is maintained, but does not go down below a prescribe quantity. To deal with the problem in fuzzy environment, an approach is proposed using possibility/necessity measure of fuzzy constraints. To solve such a real life problem in both crisp and fuzzy environments, here an ant colony algorithm is proposed and its performance is studied and compared against another soft computing algorithm GA. Numerical studies shows that both the algorithms ACO and GA are suitable for solving the proposed problem.

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Appendix A: Path of Best tour for different data set of TSPLIB

Problem: br17 for 17 Cities: By proposed ACO, path is 9, 12, 10, 7, 8, 16, 4, 3, 6,15, 5, 14, 11, 0, 2, 13, 1 and cost=39. By proposed GA, path is 9, 12, 10, 7, 8, 16, 4, 3, 6, 15, 5, 14, 11, 0, 2, 13, 1 and cost=39

Problem: ftv33 for 34 Cities: By proposed ACO, path is 0, 13, 12, 14, 15, 16, 1, 29, 26, 22, 27, 28, 25, 24, 23, 19, 20, 21, 31, 18, 17, 11, 8, 10, 9, 32, 7, 4, 6, 5, 30, 33, 2, 0 and cost=1343. By proposed GA, path is 0, 13, 25, 26, 16, 31, 30, 22, 8, 17, 7, 15, 29, 24, 14, 27, 20, 6, 10, 19, 4, 3, 28, 23, 12, 21, 9, 32, 18, 2, 5, 11, 0 and cost=1349.

Problem: ftv55 for 56 Cities: By proposed ACO, path is 0, 33, 2, 13, 35, 4, 6, 5, 47, 48, 31, 46, 29, 26, 25, 24, 42, 22, 41, 21, 50, 23, 54, 27, 49, 43, 44, 28, 53, 45, 30, 55, 34, 1, 3, 7, 32, 8, 36, 9, 37, 11, 19, 20, 40, 18, 39, 38, 10, 51, 14, 12, 15, 16, 17, 0 and cost=1635. By proposed GA path is 0, 33, 2, 13, 35, 4, 6, 5, 47, 48, 31, 46, 55, 34, 1, 3, 7, 32, 8, 36, 9, 37, 11, 19, 18, 39, 38, 10, 51, 14, 12, 15, 16, 17, 52, 26, 25,24 , 42, 21, 22, 20, 40, 41, 50, 23, 54, 27, 49, 43, 44, 28, 53, 45, 30, 0 and cost=1663.

Problem: ry48p for 48 Cities: By proposed ACO, path is 0, 8, 39, 14, 11, 32, 45, 10, 22, 12, 24, 13, 2, 21, 15, 40, 33, 28, 1, 3, 25, 34, 44, 23, 9, 41, 4, 47, 38, 31, 20, 46, 19, 16, 42, 29, 5, 26, 18, 36, 27, 17, 35, 6, 43, 30, 37, 0 and cost=14915. By proposed GA, path is 0, 8, 39, 14, 11, 32, 45, 10, 22, 12, 24, 13, 2, 21, 15, 40, 33, 28, 1, 3, 25, 34, 44, 23, 9, 41, 4, 47, 38, 31, 20, 46, 19, 16, 42, 29, 5, 26, 18, 36, 27, 17, 35, 6, 43, 30, 37, 0 and cost=14915

Problem: ft70 for 70 Cities: By proposed ACO, path is 0, 1, 60, 59, 57, 44, 48, 43, 49, 47, 33, 31, 30, 46, 45, 62, 61, 58, 27, 22, 21, 24, 56, 53, 20, 18, 19, 25, 35, 36, 42, 41, 40, 38, 39, 37, 69, 67, 63, 66, 64, 65, 68, 34, 32, 29, 12, 14, 10, 9, 7, 3, 5, 6, 28, 54, 52, 13, 8, 11, 2, 4, 23, 26, 55, 51, 50, 17, 16, 0 and cost=39702. By proposed GA path is 0, 1, 60, 59, 57, 44, 48, 43, 49, 47, 33, 31, 30, 46, 45, 62, 61, 58, 27, 22, 21, 24, 56, 53, 20, 18, 19, 25, 35, 36, 42, 41, 40, 38, 39, 37, 69, 67, 63, 66, 64, 65,68, 34, 32,29, 12, 14, 10, 9, 7, 3, 5, 6, 28, 54, 52, 13, 8, 11, 2, 4, 23, 26, 55, 51, 50, 17, 16, 0 and cost=39702.

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A COGNITIVE STYLE AND AGGREGATION OPERATOR MODEL: A LINGUISTIC APPROACH FOR CLASSIFICATION AND SELECTION OF THE AGGREGATION OPERATORS

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Abstract. Aggregation operators (AOs) have been studied by many scholars. As many AOs are proposed, there is still lacking approach to classify the categories of AO, and to select the appropriate AO within the AO candidates. In this research, each AO can be regarded as a cognitive style or individual difference. A Cognitive Style and Aggregation Operator (CSAO) model is proposed to analyze the mapping relationship between the aggregation operators and the cognitive styles represented by the decision attitudes. Four algorithms are proposed for CSAO: CSAO-1, CSAO-2 and two selection strategies on the basis of CSAO-1 and CSAO-2. The numerical examples illustrate how the choice of the aggregation operators on the basis of the decision attitudes can be determined by the selection strategies of CSAO-1 and CSAO-2. The CSAO model can be applied to decision making systems with the selection problems of the appropriate aggregation operators with consideration of the cognitive styles of the decision makers.

1. Introduction

Aggregation Operators (AOs) are applied in many domains on problems concerning the fusion of a collection of information granules. These domains include mathematics, physics, engineering, economics, business, management, and social sciences. Although the discussions of AOs are very broad [1, 3, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 19, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 40, 41], there is a lack of research about the best practice in choosing aggregation operators. The selection of the AOs can make use of the theory of cognitive style, but it seems no research has investigated the relationship between aggregation operators and the cognitive styles. Cognitive styles can be used to select the best individual for decision making.

The term 'cognitive style', was used by Allport [2], and has been described as a person's typical or habitual mode of problem solving, thinking, perceiving and remembering [22]. A style is considered to be a fairly fixed characteristic of an individual [22]. Studies in cognitive styles initially developed as a result of interest in individual differences, particularly during the 1960's [22]. Since the early 1970s, they have been more seriously considered by the teaching and training world [22]. In

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this research, the cognitive styles are associated with the development of artificial intelligence. The new motivation could be called computational cognitive style, which is to classify the individual styles of the algorithms or functions under the same objective. This research shows the classification of the aggregation operators using the cognitive styles.

Different researchers have used a variety of labels for the styles they have investigated. For example, Rading and Cheema [22] suggested that the labels be grouped into two principal cognitive styles. These were labeled the Wholist-Analytic and Verbialiser-Imager dimensions. Most researchers apply a set of uni-dimensional labels, which are postulated in the individual preferences, for the quantitative research of the cognitive style. This leads not to having a formal definition of the labels of the cognitive style. In this research, the cognitive style is described by a decision attitude variable which includes three basic linguistic labels: pessimistic, neural, and optimistic.

Many studies use linguistic methods in the decision making, for example [6, 12, 17, 18, 21, 35, 38, 39]. In fuzzy decision making process, a linguistic label is usually represented by a fuzzy number. Expert uses the linguistic label sets to access the candidates with respect to a series of criteria. An assessment result that is the selected fuzzy number assigned to the candidate under a criterion is called an information granule. The aggregation operator takes a set of information granules as input to yield a final meaningful result. There are many aggregation operators which lead to different final results. It is difficult to evaluate which aggregation operator performs better than others. This research suggested that an aggregation operator is analogue to the cognitive style of a human expert, and such cognitive style can be represented by a linguistic variable which is represented by a native fuzzy number. However, it seems that none of research discusses the use of linguistic method for such classification and selection of the aggregation operators.

The aggregation operator is a function or an algorithm to process information, analogous to the humans information process, which should be related to the scope of Cognitive psychology. Cognitive psychology is committed to using computers as a tool for aiding understanding of the mind [4]. Computational intelligence is one of studies of cognitive psychology. Cognitive style is the individual differences of the information processes of the mind. As there are similar relationships between the attributes of the aggregation operators and the cognitive styles, this paper proposes the Cognitive Style and Aggregation Operator (CSAO) model, which includes several algorithms to classify the individual styles of the AOs using the linguistic approach. In this paper, CSAO is the extension of Yuens work [36], which established the foundation for CSAO-1.

The paper is organized as follows. Section 2 defines the properties of aggregation operators whilst section 3 reviews different categories of the aggregation operators. On the basis of reviewed aggregation operators and first order linguistic ordinal scale using for cognitive style linguistic terms, section 4 proposes a CSAO-1 model. Section 5 introduces the Compound Linguistic Ordinal Scale (CLOS), which is the second order linguistic ordinal scale. Section 6 proposes a CSAO-2 model based on CLOS and CSAO-1. The numerical analyses are illustrated in section 7.

2. Fundamental Definitions of Aggregation Operators

The formal definitions of aggregation operators are as follows.

Definition 2.1. A generic aggregation operator *Agg* is a function which aggregates a set of granules $X = (x_1, \ldots, x_i, \ldots, x_n)$ into an aggregated value y. It has the form:

$$y = Agg_{(n)}^{(t)}(\alpha; (x_1, \ldots, x_i, \ldots, x_n)) = Agg_{(n)}^{(t)}(\alpha; X) \quad (1)$$

t is the length of tuple(s) of x_i and n is the number of the granules. α is a construct parameter or a bag of construct parameters to scale *Agg*.

Sometimes, α is not shown if the information of α is not important for discussion in some scenarios. Likewise, AO can be simplified as the notations such as *Agg*, $Agg(\alpha; X)$, $Agg^{(t)}(\alpha; X)$ or $Agg_{(n)}^{(t)}(\alpha; X)$. This research is only interested in $t \in \{1, 2\}$. To extend Definition 2.1, the following definition is proposed.

Definition 2.2. *Agg* is a non-weighted AO such that $x_i = c_i$ where $c_i \in C$ is a single element, or 1-tuple element. It has the form:

$$Agg^{(1)}(\alpha; X) = Agg^{(1)}(\alpha; (c_1, \ldots, c_i, \ldots, c_n)) = Agg^{(1)}(\alpha; C) \quad (2)$$

Definition 2.3. *Agg* is a weighted AO such that $x_i = \{c_i, v_i\}$ where $v_i \in V = (v_1, \ldots, v_n)$ is a utility weight. Thus x_i is a pair (or 2-tuple). The weighted AO is of the form:

$$Agg^{(2)}(\alpha; X) = Agg^{(2)}(\alpha; (\{c_1, v_1\}, \ldots, \{c_i, v_i\}, \ldots, \{c_n, v_n\})) \quad (3)$$

Definition 2.4. If $w_i = \frac{v_i}{\sum_{i=1}^{n} v_i}$, then $w_i \in W = (w_1, \ldots, w_n)$ is the probability weight such that $\sum\limits_{i \in \{1,\ldots,n\}} w_i = 1$. Thus A is a normalized weighted AO of the form:

$$Agg^{(2)}(\alpha; X) = Agg^{(2)}(\alpha; (\{c_1, w_1\}, \ldots, \{c_i, w_i\}, \ldots, \{c_n, w_n\})) \quad (4)$$

This paper focuses on discussion of normalized weighted AO.

Let y be the output of AO of X. Usually $y$and c_i have a fix interval $I' = [a, b] \subseteq [-\infty, \infty]$. Many studies used the fix interval $I = [0, 1]$ for discussion. This is the only mathematical matter of scaling or normalizing I' into I. To merge the discussion with other studies, and to associate membership theory with the aggregation problems (as the membership value also belongs to $[0, 1]$), this research uses a fix interval $I = [0, 1]$. The scaling functions of I' into I are beyond the research topic here. Now let X and y be scaled, and the extension of Definition 2.5 be as follows.

Definition 2.5. Let $c_i, y \in I$, $I = [0, 1]$. A non-weighted aggregation operator is the function$Agg : I^n \to I$. A weighted aggregation operator is the function $Agg : V^T \times I^n \to I$, and a normalized weighted aggregation operator is the function $Agg : W^T \times I^n \to I$.

According to [29, 13, 16, 8, 7] , there are some properties for the aggregators:

(1) Boundary conditions: $Agg(0, \ldots, 0) = 0$ and $Agg(1, \ldots, 1) = 1$;

(2) Monotonicity:$Agg\left(x_1,\ldots,x_i,\ldots,x_n\right)\geq Agg\left(x_1,\ldots,x_i',\ldots,x_n\right)$ if $x_i\geq x_i'$.
(3) Continuity: A is continuous with respect to each of its variables.
(4) Associativity: $Agg\left(x_1,x_2,x_3\right)=Agg\left(x_1,Agg\left(x_2,x_3\right)\right)$
$=Agg\left(Agg\left(x_1,x_2\right),x_3\right)$.
(5) Symmetry: also known as commutativity or anonymity. For every permutation δ of $\{1,2,\ldots,n\}$, the operator satisfies: $Agg\left(x_{\delta(1)},x_{\delta(2)},\ldots,x_{\delta(n)}\right)=Agg\left(x_1,x_2,\ldots,x_n\right)$.
(6) Bisymmetry: $Agg\left(A\left(x_{11},x_{12}\right),Agg\left(x_{21},x_{22}\right)\right)=$
$Agg\left(Agg\left(x_{11},x_{21}\right),Agg\left(x_{12},x_{22}\right)\right)$
(7) Absorbent Element: $Agg\left(x_1,\ldots,a,\ldots,x_n\right)=a$;
(8) Neutral Element: $Agg^{(n)}\left(x_1,\ldots,e,\ldots,x_n\right)=A^{(n-1)}\left(x_1,\ldots,x_{n-1}\right)$
(9) Idempotence: $Agg\left(x,x,\ldots,x\right)=x$;
(10) Compensation: $\min_{i=1}^{n}\left(x_i\right)\leq Agg\left(x_1,x_2,\ldots,x_n\right)\leq\max_{i=1}^{n}\left(x_i\right)$
(11) Reinforcement: full, downward, and upward reinforcements [29].

Different operators are associated with different choices of the above properties. There are no absolute rules that associate properties to operators. The researchers usually define some properties, and then create their operators.

3. **Categories of Aggregation Operators**

A non-weighted AO is the special case of a weighted AO such that all weights are equal. This study focuses on discussing the weighted AO. Aggregation operators have been contributed by many researchers. The followings introduce AOs which are frequently used and discussed.

3.1. **Quasi-linear Means.** The general form of quasi-linear means [5, 19, 23] is of the form:

$$qlm\left(W,C\right)=h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\omega_i h\left(c_i\right)\right)\quad,\quad c\in I^p. \tag{5}$$

The function $h:I\to\Re$, called the generator of $qlm\left(w,c\right)$ is continuous and strictly monotonic. If $h\left(x\right)=x^{\alpha}$, qlm is the weighted root power (wrp) or weighted generalized mean, and other three types are extensions (Table 1).

1. Weighted Root Power $wrp\left(\alpha;W,C\right)=\left(\sum\limits_{i=1}^{n}w_ic_i^{\alpha}\right)^{1/\alpha}$	2. Weighted Harmonic mean $(\alpha\to-1)$ $whm\left(W,C\right)=\frac{1}{\sum_{i=1}^{n}\frac{w_i}{c_i}}$
3. Weighted Geometric mean $(\alpha\to0)$ $wgm\left(W,C\right)=\prod\limits_{i=1}^{n}c_i^{w_i}$	4. Weighted Arithmetic mean $(\alpha\to1)$ $wam\left(W,C\right)=\sum_{i=1}^{n}w_ic_i$

Table 1. Some Forms of Quasi-linear Means

3.2. **Ordered Weighted Averaging.** OWA [29, 33] is the weighted arithmetic mean (*wam*) in which its weight values are related to the order position of C.

$$owa\,(W,C)=\sum\nolimits_{i=1}^{n}w_ib_j, \tag{6}$$

where b_j is the jth largest of the C, $w_i\in[0,1]$ and $\sum\limits_{i\in\{1,\ldots,n\}}w_i=1$. w_i can be generated from a regular non-decreasing quantifier Q, which is of the form:

$$w_i=Q\left(\frac{i}{n}\right)-Q\left(\frac{i-1}{n}\right),\quad i=1,\ldots,n, \tag{7}$$

where Q can be defined by $Q\,(\alpha;r)=r^{\alpha}$, $\alpha\geq0$.

3.3. **Weighted Median.** In weighted median aggregation [30, 23], each element c_i is replaced by two elements:

$$c_i^{+}=(1-w_i)+w_i\cdot c_i \tag{8}$$

$$c_i^{-}=w_i\cdot c_i. \tag{9}$$

Then the median value is computed by

$$wmed\,(W,C)=Median\left(c_1^{+},c_1^{-},\ldots,c_i^{+},c_i^{-},\ldots,c_n^{+},c_n^{-}\right). \tag{10}$$

Alternatively, c_i^{+} and c_i^{-} can be computed by T-conorm and T-norm, denoted as S and T respectively, having the forms as below

$$c_i^{+}=S\,(1-w_i,c_i) \tag{11}$$

$$c_i^{-}=T\,(w_i,c_i) \tag{12}$$

which S and $T$are also defined in section 3.4.

3.4. **T-norms and T-conorms.** T-norms have the properties in which $T\,(x,1)=x$ and $T\,(x,y)\leq\min\,(x,y)$ whilst T-conorms have the properties in which $S\,(x,0)=x$ and $S\,(x,y)\leq\max\,(x,y)$ [8]. Different kinds of T-norms and T-connorms [8, 23] are shown in table 2.

3.5. **Weighted Gamma Operator.** Zimmermann and Zysno [41] proposed an gamma operator on the unit interval based on T-norms and T-conorms. Calvo and Mesiar [7] modified the equation with a weighted assignment, which is of the form:

$$wgo\,(\alpha;C,W)=\left(\prod\nolimits_{i=1}^{n}c_i^{w_i}\right)^{1-\alpha}\left(1-\prod\nolimits_{i=1}^{n}(1-c_i)^{w_i}\right)^{\alpha}. \tag{13}$$

3.6. *OWMAX* **and** *OWMIN***.** Ordered weighted maximum (*owmax*) and ordered weighted minimum operators (*owmin*) were proposed by Dubois el at. [9]. Unlike OWA which deals with weighted arithmetic mean, owmax and owmin apply weighted maximum and minimum [19]. For any weight vector $W=(w_1,\ldots,w_n)\in[0,1]^n$ such that $1=w_1\geq\ldots\geq w_n$, *owmax* is of the form:

$$owmax\,(W,C)=\bigvee_{i=1}^{n}\left(w_i\wedge c_{(i)}\right),\quad C\in[0,1]^n. \tag{14}$$

For $W\in[0,1]^n$ such that $w_1\geq\ldots\geq w_n=0$, *owmin* is of the form:

$$owmin\,(W,C)=\bigwedge_{i=1}^{n}\left(w_i\vee c_{(i)}\right),C\in[0,1]^n. \tag{15}$$

1. Min-Max	$Tm\left(a,b\right)=\min\left\{a,b\right\}$
	$Sm\left(a,b\right)=\max\left\{a,b\right\}$
2. Lukasiewicz	$Tl\left(a,b\right)=\max\left\{a+b-1,0\right\}$
	$Sl\left(a,b\right)=\min\left\{a+b,1\right\}$
3. Product/ Probabilistic	$Tp\left(a,b\right)=ab$
	$Sp\left(a,b\right)=a+b-ab$
4. Dubois & Prade	$Tdp\left(\alpha;a,b\right)=\frac{a\cdot b}{\max\{a,b,\alpha\}},\quad \alpha\in\left(0,1\right)$
	$Sdp\left(\alpha;a,b\right)=1-\frac{(1-a)(1-b)}{\max\{(1-a),(1-b),\alpha\}},\alpha\in\left(0,1\right)$
5. Yager	$Ty\left(\alpha;a,b\right)=\max\left\{0,1-\left[\left(1-a\right)^{\alpha}+\left(1-b\right)^{\alpha}\right]^{1/\alpha}\right\}$
	$Sy\left(\alpha;a,b\right)=\min\left\{1,\left(a^{\alpha}+b^{\alpha}\right)^{1/\alpha}\right\},,\alpha>0$
6. Frank	$Tf\left(\alpha;a,b\right)=\log_{\alpha}\left[1+\frac{\left(\alpha^{a}-1\right)\left(\alpha^{b}-1\right)}{\alpha-1}\right],\alpha>0,\alpha\neq 1$
	$Sf\left(\alpha;a,b\right)=1-\log_{\alpha}\left[1+\frac{\left(\alpha^{1-a}-1\right)\left(\alpha^{1-b}-1\right)}{\alpha-1}\right]$
7. Weber-Sugeno	$Tws\left(\alpha_T;a,b\right)=\max\left\{\frac{a+b-1+\alpha_T\cdot a\cdot b}{1+\alpha_T},0\right\},\alpha_T>-1$
	$Sws\left(\alpha_T;a,b\right)=\min\left\{a+b+\alpha_S\cdot a\cdot b,1\right\},\alpha_S=\frac{\alpha_T}{(1+\alpha_T)}$
8. Schweizer & Sklar	$Tss\left(\alpha;a,b\right)=1-\left[\left(1-a\right)^{\alpha}+\left(1-b\right)^{\alpha}-\left(1-a\right)^{\alpha}\left(1-b\right)^{\alpha}\right]^{\frac{1}{\alpha}}$
	$Sss\left(\alpha;a,b\right)=\left[a^{\alpha}+b^{\alpha}-a^{\alpha}b^{\alpha}\right]^{\frac{1}{\alpha}},\quad \alpha>0$

TABLE 2. List of T-norms and T-connorms

3.7. **Leximin Ordering.** Leximin ordering was proposed by Dubois et al. [10]. Yager [31] improved the Lexmin ordering, based on OWA weights. Let Δ denotes a distention threshold between the values being aggregated, the Leximin is of the form:

$$leximin\left(W,C\right)=\sum_{i=1}^{n}w_ib_i \tag{16}$$

, where b_i is a sorted $C\in I^n$ in descending order such that $b_1>\ldots>b_n$. In addition,

$$w_j=LexW\left(\Delta,n\right)=\begin{cases}\frac{\Delta^{(n-j)}}{(1+\Delta)^{n-j}}\ ,j=1\\ \frac{\Delta^{(n-j)}}{(1+\Delta)^{n+1-j}},j=2,\ldots,n\end{cases},\quad w_j\in W \tag{17}$$

4. Decision Attitude and Aggregation Operator 1 (DAAO-1, or CSAO-1)

Under uncertainty, different decision makers would have different decision attitudes since they have characteristics of cognitive style or individual difference. The decision attitudes (DAs) can be described by a collection of linguistic terms represented by a collection of DA atomic fuzzy sets, $D=\{d_1,\cdots,d_j,\cdots,d_p\}$, (or the 1^{st} degree DA fuzzy variable) which is further classified as a collection of compound fuzzy sets $HD = \{d_{ij} : i = 1, \ldots, p; j = 1, \ldots, r\}$ with added directional hedge fuzzy sets $H = \{h_1, \ldots, h_r\}$ (The 2^{nd} degree DA fuzzy variable). Details of compound fuzzy variable are shown in section 5.

The range of the membership of a decision attitude fuzzy set is in [0,1] and the aggregated value also belongs to [0,1]. The aggregated value of the membership (or the likelihood) of a decision attitude has the relationship, shown in the following definition.

Definition 4.1. An aggregated value y from a normalized aggregation operator Agg of the set of input parameters X belongs to a decision attitude fuzzy set d_j, with the membership value $d_j(y) \in [0, 1]$ by the membership function $d_j : y \to I$, $I = [0, 1]$.

As the fuzzy set is characterized by the membership function, the same notation d_j is used for a fuzzy set of membership. Usually, the membership function applies a triangular function $\mu_j(a, b, c)$ which is defined by three points.

Different input parameter sets, X's, result in different Effective Aggregation Ranges (EAR) from a collection of the aggregation operators. The effective aggregation range $[y_*, y*]$ is defined as follows.

Definition 4.2. Let the set of the aggregated values from the set of the aggregation operators $\widetilde{Agg}$ be $Y = (y_1, \ldots, y_k, \ldots, y_m)$. The permutation of Y is $\vec{Y} = \{y_{(1)}, \ldots, y_{(k)}, \cdots, y_{(m)}\}$, where $y_{(1)} \leq y_{(2)} \cdots \leq y_{(m)}$. Thus, the Effective Aggregation Range is $[y_*, y*]$, where $y_* = y_{(1)} = \min(Y)$ is the low-boundary, and $y* = y_{(m)} = \max(Y)$ is the up-boundary.

Lemma 4.3. *The EAR is the proper subset of I, i.e. $[y_*, y*] \subseteq [0, 1]$ (see Figure 1).*

Proof. As $y = A_{(n)}^{(t)}(\alpha; X) \in I = [0, 1]$, $y_* = \min(Y) \geq 0$, and $y* = \max(Y) \leq 1$, the lemma holds. □

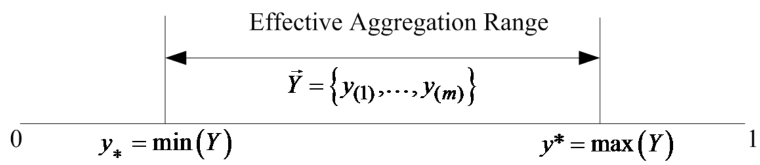

Figure 1. Effective Aggregation Range of AOs

Lemma 4.4. *The collection of AOs is the form $\widetilde{Agg} : X \to [y_*, y*]^m$, where m is the dimension (the number) of the output set.*

Proof. This lemma is directly derived from Definition 4.2. □

The CSAO model describes how the cognitive styles of the aggregation operators can be reflected by the decision attitudes. The CSAO can be represented by a collection of the DA fuzzy sets. Thus, the following proposition holds.

Proposition 4.5. *(D_{Agg}): The collection of decision attitude fuzzy sets for an aggregation operator A is of the form:*

$$D_{Agg} = \{\{y, d_1(y)\}, \ldots, \{y, d_j(y)\}, \ldots, \{y, d_p(y)\}\}, y \in [y_*, y*] \tag{18}$$

Proof. Let the collection of decision attitude fuzzy sets be $D = \{d_1, \cdots, d_j, \cdots, d_p\}$, and the discourse universal of D be $[y_*, y*] \subseteq [0, 1]$ (Lemma 4.3). Thus the collection of memberships of the set of decision attitudes D for an aggregation operator is $D_A : [y_*, y*] \rightarrow I^p$. As the fuzzy set is generally defined as a collection of pairs, the form is given above. □

Proposition 4.6. **($D_{\widetilde{Agg}}$):** *A collection of the 1^{st} degree DA fuzzy sets $D_{\widetilde{Agg}}$ for a collection of aggregation operators $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_m)$ is of the form:*

$$D_{\widetilde{Agg}} = \begin{pmatrix} \{\{y_{(1)}, d_1\} & ,\ldots, & \{y_{(1)}, d_j\} & ,\ldots, & \{y_{(1)}, d_p\}\} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \{\{y_{(k)}, d_1\} & ,\ldots, & \{y_{(k)}, d_j\} & ,\ldots, & \{y_{(k)}, d_p\}\} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \{\{y_{(m)}, d_1\} & ,\ldots, & \{y_{(m)}, d_j\} & ,\ldots, & \{y_{(m)}, d_p\}\} \end{pmatrix}, \tag{19}$$

where $\{y_{(k)}, d_j\} = \{y_{(k)}, d_j(y_{(k)})\}, \forall k, \forall j,$ *and* $y_{(1)} \le y_{(2)} \cdots \le y_{(m)}$.

Proof. It follows from Proposition 4.5 and Lemma 4.4. □

Definition 4.7. The Information Fusion Process $IFP = (\bar{X}, Y, AO*, \{AO\}, SAO)$ is the function which aggregates multiple sources of data granules $\bar{X}$ as a meaningful value Y to represent an object by selecting the most appropriate aggregation operator (AO*) among a set of the AO candidates $\widetilde{AO} = \{AO\}$, i.e. $SAO : \{AO\} \rightarrow AO^*$, and $AO^* : \bar{X} \rightarrow Y$.

The CSAO model is the ideal function for SAO. Following of the above definition, two definitions are proposed for the selection of AO in $D_{\widetilde{Agg}}$.

Definition 4.8. If an aggregation operator has more than one membership of DAs, the selection of DAs for the AO is of the form:

$$d^*(k) = ArgMax\left(\{\{y_{(k)}, d_1\}, \ldots, \{y_{(k)}, d_j\}, \ldots, \{y_{(k)}, d_p\}\}\right) \tag{20}$$

Definition 4.9. If a DA linguistic term includes more than one aggregation operator, the selection of AOs in a DA linguistic term is of the form:

$$d_j^* = ArgMax\left(\{\{y_{(1)}, d_j\}, \ldots, \{y_{(k)}, d_j\}, \ldots, \{y_{(m)}, d_j\}\}\right) \tag{21}$$

The *DAAO-1* for CSAO is in the following algorithm.

Algorithm 4.10. $DAAO\text{-}1 = CSAO\text{-}1\left(D, \widetilde{Agg}, X\right)$:

Input:

a. A collection of the membership functions of DA fuzzy sets $D = \{d_1, \cdots, d_j, \cdots, d_p\}$;

b. A collection of AOs: $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_m)$;
c. A collection of information granules: $X = (x_1, \ldots, x_i, \ldots, x_n)$;

Process:

Step 1: Compute $\widetilde{Agg}\,(X)$, and then $Y = (y_1, \ldots, y_k, \ldots, y_m)$ is achieved;
Step 2: Get the permutation of Y: $\vec{Y} = \left\{y_{(1)}, \ldots, y_{(k)}, \cdots, y_{(m)}\right\}$;
Step 3: Get $[y_*, y^*] = \left[y_{(1)}, y_{(m)}\right]$;
Step 4: Calculate intervals and modal values for $D$by equally dividing $[y_*, y^*]$;
i. $d_1 = \left(y_*, y_*, y_* + \frac{y^*-y_*}{p-1}\right)$
ii. $d_{j\neq 1,p} = \left(y_* + \frac{y^*-y_*}{p-1}(j-2), y_* + \frac{y^*-y_*}{p-1}(j-1), y_* + \frac{y^*-y_*}{p-1}(j)\right)$
iii. $d_p = \left(y^* - \frac{y^*-y_*}{p-1}, y^*, y^*\right)$
Step 5: Elicit memberships for $D$by interpolation of the three points (a,b,c):
Step 6: Calculate $D\left(\vec{Y}\right)$, $D_{\widetilde{Agg}}$ and $d^*\,(k)$, $\forall k$.
Step 7: Get $d_j^*, \forall j$.

Output:. $\left\{d_j^*\right\}$. **//END**

This study focuses on discussion of the weighted aggregation operators of which $x_i = \{w_i, c_i\} \in X$is the input.

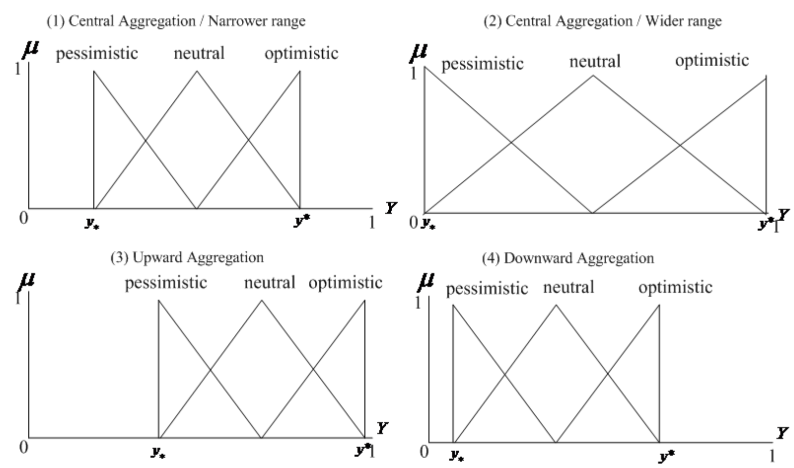

FIGURE 2. Properties of Effective Aggregation Range

To conclude, the CSAO description model is the function $g : X \to I$ or $g = \widetilde{Agg} \circ D = D\left(\widetilde{Agg}\,(X)\right)$. It means that the function g maps the collection of information granules X with the set of the aggregators $\widetilde{Agg}$, to the membership interval [0,1] corresponding to the collection of decision attitude fuzzy sets D.

In most practice, the decision attitudes can be described by three linguistic terms: pessimistic, neutral and optimistic. Figure 2 shows some properties of the DA fuzzy sets.

Properties of EAR can be summarized as followings.

Proposition 4.11. *Let* $y' = mean\,(y_*, y*) = \frac{1}{2}\,(y_* + y*)$*, then*

(1) *Effective aggregation range (EAR) is downward aggregation if* $y' < 0.5$*;.*
(2) *EAR is upward aggregation if* $y' > 0.5$*;*
(3) *EAR is central aggregation if* $y' = 0.5$*;*
(4) *EAR 2 is more upward than EAR 1 if* $y'_1 < y'_2$*. Or EAR 1 is more downward than the EAR 2.*
(5) *EAR 2 is wider than EAR 1 if* $y*_1 - y_{*_1} < y*_2 - y_{*_2}$*. Or EAR 1 is narrower than EAR 2.*

Example 4.12. A numerical example analysis of the algorithm of the CSAO-1 description model is illustrated as follows.

Input

a. Define the collection of decision attitude fuzzy sets:
Let $D = \{d_1, d_2, d_3\}$ represent the set of pessimistic, neural, and optimistic decision attitudes. $d_1 = \mu\,(y_*, y_*, y')$, $d_2 = \mu\,(y_*, y', y*)$, $d_2 = \mu\,(y', y', y*)$, where μ is the triangular membership function.

b. Define a collection of the Aggregation Operators:

$$\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_{17})$$
$$= \left\{ \begin{array}{l} wrp, whm, wgm, wam, owa, ow\max, owmin, \\ Lex\min, wgo, wmed, wmed_l, wmed_{mm}, \\ wmed_{dp}, wmed_y, wmed_f, wmed_{ws}, wmed_{ss} \end{array} \right\}$$

The aggregation operator can be found in section 3. For the notation, $wmed_l$ is $wmed$with Lukasiewicz T-norm and T-connorm. This naming convention is also applied to other $wmed$s, taking different T-norms and T-connorms. In addition, as α affects the aggregation result, different value of α can be regarded as a different operator. This example takes $\alpha = 0.2$, for all parametric operators.

c. Get the collection of information granules:
Let $X = (x_1, \ldots, x_5)$ be weighted criteria; $C = (0.4, 0.5, 0.6, 0.7, 0.9)$, $W = owaW\,(0.6, 5) = (0.3801, 0.1964, 0.1589, 0.1387, 0.1253)$,
and thus $X = ((0.4, 0.1978)\,, \ldots, (0.9, 0.6250))$.

Process

Step 1: Compute$Y$ by$\tilde{A}\,(X)$:

$$Y = \tilde{A}\,(X) = \left\{ \begin{array}{l} 0.5375, 0.5137, 0.5332, 0.5557, 0.6949, 0.3807, 0.4, 0.6939, \\ 0.5019, 0.4619, 0.5127, 0.5, 0.5, 0.5, 0.1193, 0.5199, 0.4868 \end{array} \right\}$$

(k)	Agg	$y_{(k)}$	$D\left(y_{(k)}\right)$	$d^*(k)$
1	$wmed_f$	0.1193	{1,0,0}	Pess
2	$ow\max$	0.3807	{0.0917,0.9083,0}	Pess
3	$ow\min$	0.4	{0.0247,0.9753,0}	Ntl
4	$wmed$	0.4619	{0,0.8095,0.1905}	Ntl
5	$wmed_{ss}$	0.4868	{0,0.7231,0.2769}	Ntl
6	$wmed_{mm}$	0.5	{0,0.6773,0.3227}	Ntl
7	$wmed_{dp}$	0.5	{0,0.6773,0.3227}	Ntl
8	$wmed_y$	0.5	{0,0.6773,0.3227}	Ntl
9	wgo	0.5019	{0,0.6707,0.3293}	Ntl
10	$wmed_l$	0.5127	{0,0.6333,0.3667}	Ntl
11	whm	0.5137	{0,0.6298,0.3703}	Ntl
12	$wmed_{ws}$	0.5199	{0,0.6080,0.3920}	Ntl
13	wgm	0.5332	{0,0.5618,0.4382}	Ntl
14	wrp	0.5375	{0,0.5470,0.4530}	Ntl
15	wam	0.5557	{0,0.4838,0.5162}	Opt
16	owa	0.6949	{0,0,1}	Opt
17	Leximin	0.6949	{0,0,1}	Opt

TABLE 3. The Results for $D_{\widetilde{Agg}}$ of 17 AOs

Step 2**:** Get the $\vec{Y}$:

Get$Ordering\,(Y) = \{14,11,13,15,16,2,3,16,9,4,10,6,6,6,1,12,5\}$, then

$$\vec{Y} = \left\{ \begin{array}{l} 0.1193, 0.3807, 0.4, 0.4619, 0.4868, 0.5, 0.5, 0.5, 0.5019, 0.5127, \\ 0.5137, 0.5199, 0.5332, 0.5375, 0.5557, 0.6939, 0.6949 \end{array} \right\}.$$

Step 3**:** $[y_*, y*] = \left[y_{(1)}, y_{(m)}\right] = [0.1193, 0.6949]$.

Step 4 **and** 5**:** Assign intervals and interpolate memberships for D.

Let $(y_*, y', y*) = [0.1193, 0.4071, 0.6949]$ be substituted by $\mu\,(a, b, c)$ in D. CSAO-1 pattern is shown in Figure 3. It can be observed that the proposed numerical integration is downward integration as$y' = 0.4071 < 0.5$.

Step 6**:** Calculate $D\left(\vec{Y}\right)$, $D_{\widetilde{Agg}}$ and $d^*(k)$.

Table 3 summarizes the results for$D_{\widetilde{Agg}}$, $\left\{y_{(k)}, D_{Agg}\left(y_{(k)}\right)\right\} \in D_{\widetilde{Agg}}$, $\forall k \in \{1, \cdots, 17\}$.

Step 7 **and Output:** $\left\{d_j^*\right\} = \{1, 3, 17\}$, which means $\{wmed_f$, $owmin$, $owa/Leximin\}$, where owa and $Leximin$ produce the same result.

The interpretation of the above example is as follows. The weighted median with other t-connorms and t-norms [23, 30] is likely to produce questionable results. Firstly, t-conform and t-norm are initially designed for aggregation of two fuzzy sets, and are not suitable for weighted criteria, since $wmed\,(W, C)$ has different meanings for $wmed\,(C, W)$. Secondly, the definition of the tuning parameter α

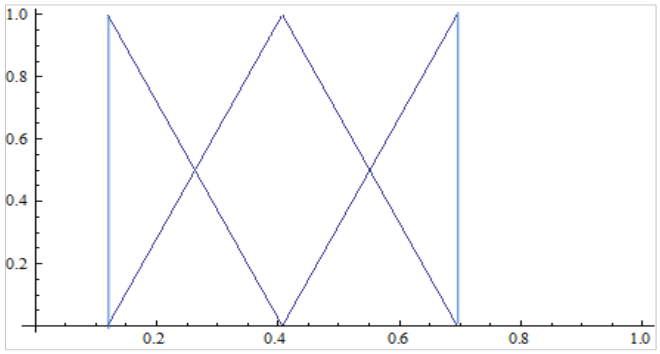

FIGURE 3. Fuzzy Sets in CSAO-1 Pattern

is infinitive since each α represents a new aggregation operator due to different output values. Thirdly, the more criteria are aggregated, the lesser values in W *as* $\sum_{w_i \in W} w_i = 1$ are followed. As t-norms or t-conorms is mainly based on Min and Max of two sets, a misleading result will result.

owmax and *owmin* are not the effective AOs for the decision matrix. The third reason of the above description explains this issue. *Lexmin* and *owa* produce the same result as the weights used by them, and are not defined by their intrinsic functions. If these aggregation operators are removed, the new result is shown in Example 4.13. Further investigation for *owa* is concluded after illustration of Example 4.13.

Example 4.13. Let be $\widetilde{Agg} = \{Agg_1, \ldots, Agg_k, \ldots, Agg_7\}$ ={*wrp,whm,wgm,wam, owa,wgo,wmed*}. Others remain unchanged. The new results for $D_{\widetilde{Agg}}$ are shown in Table 4, and finally, $\{d_j^*\} = \{1, 6, 7\}$, which is {*wmed*, *wam*, *owa*}.

(k)	Agg	$y_{(k)}$	$D_{\widetilde{Agg}}(y_{(k)})$	$d^*(k)$
1	*wmed*	0.4619	{1,0,0}	Pess
2	*wgo*	0.5019	{.6570,0343,0}	Pess
3	*whm*	0.5137	{.5558,04442,0}	Pess
4	*wgm*	0.5332	{.3880,06120,0}	Ntl
5	*wrp*	0.5375	{.3513,06487,0}	Ntl
6	*wam*	0.5557	{.1953,08047,0}	Ntl
7	*owa*	0.6949	{0,0,1}	Opt

TABLE 4. The Results for $D_{\widetilde{Agg}}$ of Seven AOs

Examples 4.12 and 4.13 imply that not all AOs can be applied in DSAO (DAAO). It is similar to that not all the people being suitable for a single job, an interest, or a subject domain as they have different cognitive styles. People who are suitable for a job are pooled and selected accordingly with respect to the senior decision maker. Thus only the suitable AOs can be taken in DSAO, and then classified. The one which mostly reflects the decision maker's cognitive style is selected.

In addition, *owa* seems to produce exaggerate results in the above example. The main reason is that the order of the values of the criteria is sorted in descending order. This action is unnecessary. For one reason, the weight and the criterion are matched; for another reason, the different initial settings of the criteria order are very likely to produce different results. For the third reason, there is no point to mismatch the weight and the criterion pair.

The next section discusses CLOS prior to discuss DSAO-2 which is based on CLOS and CSAO-1.

5. **Compound Linguistic Ordinal Scale**

Compound Linguistic Ordinal Scale (CLOS) and its application was developed by Yuen [37]. CLOS is a Deductive Rating Strategy (Rs) of the Hedge-Direction-Atom Linguistic Representation Model (HDA-LRM) with a cross reference relationship.

In the HDA-LRM, Compound Linguistic Variable (CLV) $\aleph$, a matrix of a large number of linguistic descriptors is produced by the syntactic rule. The semantic rule "Computing with CLV" maps CLV into representation numbers in matrix $\overline{X}_{\aleph}$ or $\overline{X}$ by Fuzzy Normal Distribution $f_{\overline{X}}(\aleph)$, and produces the numerical results meeting the different requirements of different scenario using few scalable descriptable user-defined parameters.

The Deductive Rating Strategy (Rs) is the ideal rating interface for handing the large scale of CLV. Three key concepts are presented, as follows.

5.1. **Syntactic Rule.** Regarding the syntactic form, CLOS is established on a compound linguistic variable $\alpha \in \aleph_{mn}$ which is comprised of the elements from the linguistic term vectors respectively: hedge vector$\overrightarrow{V_h}$ directional vector $\overrightarrow{V_d} = \left[v_d^-,\ v_d^\theta,\ v_d^+\right]$ and atomic vector $\overrightarrow{V_a} = \left[v_{a_j}\right]$. A matrix of Compound Linguistic Variable (CLV) $\aleph_{mn}$ is built on the syntactic rule algorithm (algorithm 2), $\aleph_{mn} = G_\aleph\left(\overrightarrow{V_h}, \overrightarrow{V_d}, \overrightarrow{V_a}\right)$, and has the following form:

$$\begin{bmatrix} \emptyset & v_{hd_1}\oplus v_{a_2} & \cdots & v_{hd_1}\oplus v_{a_{n-1}} & v_{hd_1}\oplus v_{a_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \emptyset & v_{hd_\eta}\oplus v_{a_2} & \ddots & v_{hd_\eta}\oplus v_{a_{n-1}} & v_{hd_\eta}\oplus v_{a_n} \\ {v_{a_1}}^\theta & {v_{a_2}}^\theta & \ddots & {v_{a_{n-1}}}^\theta & {v_{a_n}}^\theta \\ v_{hd_{\eta+2}}\oplus v_{a_1} & v_{hd_{\eta+2}}\oplus v_{a_2} & \ddots & v_{hd_{\eta+2}}\oplus v_{a_{n-1}} & \emptyset \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{hd_m}\oplus v_{a_1} & v_{hd_m}\oplus v_{a_2} & \cdots & v_{hd_m}\oplus v_{a_{n-1}} & \emptyset \end{bmatrix} \tag{22}$$

, where v_{hd} is the element of the combination of $\overrightarrow{V_h}$ and $\overrightarrow{V_d}$.

Algorithm 5.1. (Syntactic Rule Algorithm $\aleph_{mn} = G_\aleph\left(\overrightarrow{V_h}, \overrightarrow{V_d}, \overrightarrow{V_a}\right)$:

1. Input: Linguistic term sets ($\overrightarrow{V_h}$,$\overrightarrow{V_d}$,$\overrightarrow{V_a}$)

2. Proceed $G_{\overrightarrow{V_{hd}}}\left(\overrightarrow{V_h},\overrightarrow{V_d}\right)=\overrightarrow{V_{hd}}$ by

$$[v_{hd_i}]_{i=1}^{m}=\left[\left(v_{h_\eta}\oplus v_d^-\right),\ \cdots\ ,\left(v_{h_1}\oplus v_d^-\right),\ v_d^\theta\ ,\left(v_{h_1}\oplus v_d^+\right),\ \cdots\ ,\left(v_{h_\eta}\oplus v_d^+\right)\right]$$

3. Proceed $G_\aleph\left(\overrightarrow{V_{hd}},\overrightarrow{V_a}\right)$ by

$$G_{\alpha_{ij}}\left(v_{hd_i},v_{a_j}\right)\triangleq\begin{cases}\emptyset & j=1\&i\in\{1,\cdots,((m+1)/2)\}\\ v_{hd_i}\oplus v_{a_j} & j\neq 1,n\&\forall i\\ \emptyset & j=n\&i\in\{((m+1)/2),\cdots,m\}\end{cases},\forall i,j$$

4. Return: $\aleph_{mn}=G_\aleph\left(\overrightarrow{V_{hd}},\overrightarrow{V_a}\right)$ **//END**

5.2. **Semantic Rule.** The numerical representation is derived by the semantic rule algorithm or Fuzzy Normal Distribution of the form:

$$\bar{X}_\aleph=f_{\overline{X}}(\aleph)=f_{\overline{X}}\left(\left\{\left(\gamma_{\alpha^j},d_{\alpha^j},\tau_{\alpha^j},\{\mu_{\alpha^j\varphi}^{-1}\}^\varphi\right)\right\},[X_{\min},X_{\max}],\left(\phi\left(\overrightarrow{V_h}\right),\lambda_0\right)\right) \tag{23}$$

$\bar{X}_\aleph$ is the numerical representation of $\aleph$in either fuzzy or crisp value as crisp value is the special case of the fuzzy value. γ_{α^j} is the modal value, d_{α^j} is symmetric distance (by default, $d_{\alpha^1}=d_{\alpha^2}=,\ldots,=d_{\alpha^n}$), τ_{α^j} is tuning parameter of the membership function, μ_{α^j}, of α^j, and $\mu_{\alpha^j\varphi}^{-1}$ is the inversed membership function, which the default setting is the inversed parabola-based membership function $PbMF_{\alpha^j}^{-1}$ is of the form:

$$PbMF_{\alpha^j}^{-1}\left(\mu_{\alpha^j\phi}\right)=\begin{cases}\gamma_{\alpha^j}-d_{\alpha^j}\sqrt{1-\left(\mu_{\alpha^j\phi}\right)^{1/\tau_{\alpha^j}}},\phi='-'\\ \gamma_{\alpha^j},\phi='\theta'\\ \gamma_{\alpha^j}+d_{\alpha^j}\sqrt{1-\left(\mu_{\alpha^j\phi}\right)^{1/\tau_{\alpha^j}}},\phi='+'\end{cases} \tag{24}$$

, where $\varphi='-','\theta''+'$ is determined from $\overrightarrow{V_d}$

$[X_{\min},X_{\max}]$ is the interval of numerical representation of the scale. The 2-tuple input $\left(\phi\left(\overrightarrow{V_h}\right),\lambda_0\right)$ determines the distribution of the $\overrightarrow{V_{hd}}$ in the membership fuzziness process (MFI). Thus $f_{\overline{X}}(\aleph)$ is shown in Algorithm 5.2.

Algorithm 5.2. (Semantic Rule Algorithm / Fuzzy Normal Distribution):

1. Get valid $\left(\left\{\left(\gamma_{\alpha^j},d_{\alpha^j},\tau_{\alpha^j},\{\mu_{\alpha^j\varphi}^{-1}\}^\varphi\right)\right\},[X_{\min},X_{\max}],\left(\phi\left(\overrightarrow{V_h}\right),\lambda_0\right)\right)$.
2. Calculate $MCI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)$ and $MFI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)$ by

$$MFI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)=MFI\left(\begin{bmatrix}v_{h_1}\\ \vdots\\ v_{h_\eta}\end{bmatrix}\right)$$

$$=\begin{bmatrix}\left[\mu'_{l_i}-\lambda_{\mu_{l_i}}dis\left(v_{h_i}\right),1\right]_{i=1}\\ \left[\left[\mu'_{l_i}-\lambda_{\mu_{l_i}}dis\left(v_{h_i}\right),\mu'_{u_i}+\lambda_{\mu_{l_i}}dis\left(\sigma_i\right)\right]\right]_{i=2}^{\eta-1}\\ \left[0,\mu'_{u_i}+\lambda_{\mu_{l_i}}dis\left(\sigma_i\right)\right]_{i=\eta}\end{bmatrix},$$

where $dis\left(v_{h_i}\right)=\frac{\phi\left(v_{h_i}\right)}{\sum_{\overrightarrow{V_h}}\phi\left(v_{h_i}\right)}$; $\lambda_{\mu_{u_i}}=\lambda_{\mu_{l_i}}=\frac{\lambda_0}{2}$, where$\lambda_{\mu_{u_i}},\lambda_{\mu_{l_i}}\in[0,1]$ (i.e. $\lambda_0\in[0,2]$) such that $0\le\mu_{u_i}\le1$.

3. Calculate $MFI\left(\left[\left[\overrightarrow{V_h}^{+}\right]\right]\right)$ and $MFI\left(\left[\left[\overrightarrow{V_h}^{-}\right]\right]\right)$ by

$$MFI\left(\left[\left[\overrightarrow{V_h}^{-}\right]\right]\right)=vip\left(MFI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)\right)\equiv\overset{\mu_L^-\ \mu_U^-}{\left\lceil\mu_{l_j},\mu_{u_j}\right\rceil_{j=1}^{\eta}}=\overset{\mu_L\quad\mu_U}{\left[\left[\mu_{l_j},\mu_{u_j}\right]\right]_{j=\eta}^{1}}$$

$$MFI\left(\left[\left[\overrightarrow{V_h}^{+}\right]\right]\right)=hrp\left(MFI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)\right)\equiv\overset{\mu_L^+\ \mu_U^+}{\left\lfloor\mu_{l_j},\mu_{u_j}\right\rfloor_{j=1}^{\eta}}=\overset{\mu_U\quad\mu_L}{\left[\left[\mu_{U_j},\mu_{l_j}\right]\right]_{j=1}^{\eta}}$$

$$\text{,where } MFI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)=MFI\left(\begin{bmatrix}v_{h_1}\\ \vdots\\ v_{h_\eta}\end{bmatrix}\right)=\overset{\mu_L\quad\mu_U}{\left[\left[\mu_{l_j},\mu_{u_j}\right]\right]_{j=1}^{\eta}}$$

4. Calculate $FI\left(\left[\left[\widehat{\alpha^j}\right]\right]\right)$,$\forall j$ by

$$FI\left(\left[\left[\widehat{\alpha^j}\right]\right]\right)=\begin{cases}\begin{bmatrix}\left[\left[\mu^{-1}_{\alpha^{j-}}\left(\mu_{l_i}{}^{-}\right),\mu^{-1}_{\alpha^{j-}}\left(\mu_{l_i}{}^{-}\right)\right]\right]_{i=1}^{\eta}\\ \left[\left[\mu^{-1}_{\alpha^{j}}\left(\mu_{l_\theta}\right),\mu^{-1}_{\alpha^{j}}\left(\mu_{l_\theta}\right)\right]\right]_{\mathrm{i}=\eta+1}\end{bmatrix},j=1\\ \begin{bmatrix}\left[\left[\mu^{-1}_{\alpha^{j-}}\left(\mu_{l_i}{}^{-}\right),\mu^{-1}_{\alpha^{j-}}\left(\mu_{l_i}{}^{-}\right)\right]\right]_{i=1}^{\eta}\\ \left[\left[\mu^{-1}_{\alpha^{j}}\left(\mu_{l_\theta}\right),\mu^{-1}_{\alpha^{j}}\left(\mu_{l_\theta}\right)\right]\right]_{\mathrm{i}=\eta+1}\\ \left[\left[\mu^{-1}_{\alpha^{j+}}\left(\mu_{l_i}{}^{+}\right),\mu^{-1}_{\alpha^{j+}}\left(\mu_{l_i}{}^{+}\right)\right]\right]_{i=\eta+2}^{m}\end{bmatrix},1<\mathrm{j}<\mathrm{n}\quad,\forall j\\ \begin{bmatrix}\left[\left[\mu^{-1}_{\alpha^{j}}\left(\mu_{l_\theta}\right),\mu^{-1}_{\alpha^{j}}\left(\mu_{l_\theta}\right)\right]\right]_{\mathrm{i}=\eta+1}\\ \left[\left[\mu^{-1}_{\alpha^{j+}}\left(\mu_{l_i}{}^{+}\right),\mu^{-1}_{\alpha^{j+}}\left(\mu_{l_i}{}^{+}\right)\right]\right]_{i=\eta+2}^{m}\end{bmatrix},j=n\end{cases}$$

5. If $\overline{X}_{\aleph}$ is in fuzzy number, then
$\overline{X}=\left[\left[\left[\left(\bar{x}_{l_{ij}},\bar{x}_{\pi_{ij}},\bar{x}_{u_{ij}}\right)\right]\right]_{i=1}^{m}\right]_{j=1}^{n}$,
$\bar{x}_{\pi_{ij}}\in mean\left(FI\left(\left[\left[\widehat{\alpha^j}\right]\right]\right)\right)$ and $\left(\bar{x}_{l_{ij}},\bar{x}_{u_{ij}}\right)\in FI\left(\left[\left[\widehat{\alpha^j}\right]\right]\right)$, $\forall i,j$
6. Return $\overline{X}$;.

//END

5.3. **Deductive Rating Strategy.** It seems incredible that an expert can handle $|\aleph_{7\pm2,7\pm2}|=[21,73]$ linguistic terms although CLV can produce a large scale of compound linguistic terms. Thus deductive rating strategy is proposed. Algorithm 5.3 shows the rating steps whilst Figure 4 shows an example of the rating interface.

Algorithm 5.3. (Deductive Rating Strategy $\left(\overrightarrow{V_{hdj}},\overrightarrow{V_a},Rs\right)$):

1. Observe external information;
2. Understand the problem;
3. Understand the CLOS model;
4. First step rating: choose v_{a_j} in $\overrightarrow{V_a}\equiv\left[v_{a_j}\right]_{j=1}^{n}$;
5. Computer shows second options by

$$\overrightarrow{V_{hdj}} = Rs\left(v_{a_j}\right) = \begin{cases} \left[v_{hd_i}\right]_{i=1}^{\eta} & \text{if } j = 1 \\ \left[v_{hd_i}\right]_{i=1}^{m} & \text{if } j \neq 1, n \\ \left[v_{hd_i}\right]_{i=\eta+2}^{m} & \text{if } j = n \end{cases} .$$

6. Rethink the second option and revise first option;
   6.1 If first option is confirmed, then the rater chooses v_{hd_i} in $\overrightarrow{V_{hdj}}$;
   6.2 Else go to Step 3
7. Return $\alpha_{ij} = \left(v_{hd_i}, v_{a_j}\right)$ **//END**

6. Decision Attitude and Aggregation Operator 2 (DAAO-2, or CSAO-2)

Usually a fuzzy set consists of several AOs. If a decision maker chooses a linguistic term for the decision attitude, although the choices are narrowed, he still needs to choose the right one representing his cognitive style. Thus the DA atomic fuzzy set is further classified by a vector of hedge terms$H = \{h_1, \ldots, h_\eta, \ldots, h_r\}$, which is represented by a vector of the memberships of DA d_j , the following proposition holds.

Proposition 6.1. **($\{d_{ij}\} = f_{\overline{X}}(HD)$):** *The Linguistic Cartesian Product $G_\aleph$ of D and H forms a collection of compound fuzzy sets $HD = \{h_i \oplus d_j : i = 1, \ldots, r; j = 1, \ldots, p\}$, which is of the form.*

$$HD = G_\aleph(H, D) = \begin{bmatrix} \emptyset & h_1 \oplus d_2 & \cdots & h_1 \oplus d_p \\ \vdots & \vdots & \ddots & \vdots \\ \emptyset & h_\eta \oplus d_2 & \ddots & h_\eta \oplus d_p \\ {d_1}^\theta & {d_2}^\theta & \ddots & {d_p}^\theta \\ h_{\eta+2} \oplus d_1 & h_{\eta+2} \oplus d_2 & \ddots & \emptyset \\ \vdots & \vdots & \ddots & \vdots \\ h_r \oplus d_1 & h_r \oplus d_2 & \cdots & \emptyset \end{bmatrix} \tag{25}$$

Let $\{d_{ij}\}$ be the matrix of the fuzzy numbers of HD. $\{d_{ij}\}$ is determined by the semantic rule algorithm $f_{\overline{X}}(HD)$ (algorithm 5.2), which is of the form:

$$\{d_{ij} : i = 1, \ldots, r; j = 1, \ldots, p\} = f_{\overline{X}}(HD)$$

$$= f_{\overline{X}}\left(\left\{\left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \left\{\mu_{d^{j\phi}}{}^{-1}\right\}^{\phi}\right)\right\}, [y_*, y^*], \left(\varphi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right) \tag{26}$$

,where $\left\{\left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \{\mu_{d^{j\varphi}}^{-1}\}^{\varphi}\right)\right\}$ is the 1st degree DA fuzzy sets which are the symmetric fuzzy set: γ_{d^j} is the modal value, Δ_{d^j} is symmetric distance (by default, $\Delta_{d^1} = \Delta_{d^2} =, \ldots, = \Delta_{d^p}$), τ_{d^j} is the tuning parameter of the membership function, μ_{d^j} is the membership function of d_j or d^j , and $\mu_{d^{j\varphi}}^{-1}$ is the inverse membership function. The collection of the 1st degree DA fuzzy sets is called the 1st degree DA

fuzzy variable. The parameters of the membership fuzziness process $\left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)$ *determine the distribution of the* 2^{nd} *degree DA fuzzy variable with respect to the corresponding* 1^{st} *degree DA fuzzy sets.*

Proof. It follows from Algorithms 5.1 and 5.2. □

The compound linguistic terms for the decision attitude are used by a deductive rating strategy which is the double step rating process (algorithm 4). The collection of the 2^{nd} degree DA fuzzy sets is shown in the following proposition.

Proposition 6.2. ($D"_{\widetilde{Agg}}$): *A collection of the* 2^{nd} *degree DA fuzzy sets* $D"_{\widetilde{Agg}}$ *for a collection of aggregation operators* $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_m)$*is of the form:* $D_{\widetilde{Agg}}\left(\vec{Y}\right) =$

$$\begin{bmatrix} \emptyset & \left\{\begin{matrix}\{y_{(1)}, d_{1,2}\}\\ \vdots \\ \{y_{(m)}, d_{1,2}\}\end{matrix}\right\} & \cdots & \left\{\begin{matrix}\{y_{(1)}, d_{1,p}\}\\ \vdots \\ \{y_{(m)}, d_{1,p}\}\end{matrix}\right\} \\ \vdots & \vdots & \ddots & \vdots \\ \emptyset & \left\{\begin{matrix}\{y_{(1)}, d_{\eta,2}\}\\ \vdots \\ \{y_{(m)}, d_{\eta,2}\}\end{matrix}\right\} & \ddots & \left\{\begin{matrix}\{y_{(1)}, d_{\eta,p}\}\\ \vdots \\ \{y_{(m)}, d_{\eta,p}\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{y_{(1)}, d_{\eta+1,1}\}\\ \vdots \\ \{y_{(m)}, d_{\eta+1,1}\}\end{matrix}\right\} & \left\{\begin{matrix}\{y_{(1)}, d_{\eta+1,2}\}\\ \vdots \\ \{y_{(m)}, d_{\eta+1,2}\}\end{matrix}\right\} & \ddots & \left\{\begin{matrix}\{y_{(1)}, d_{\eta+1,p}\}\\ \vdots \\ \{y_{(m)}, d_{\eta+1,p}\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{y_{(1)}, d_{\eta+2,1}\}\\ \vdots \\ \{y_{(m)}, d_{\eta+2,1}\}\end{matrix}\right\} & \left\{\begin{matrix}\{y_{(1)}, d_{\eta+2,2}\}\\ \vdots \\ \{y_{(m)}, d_{\eta+2,2}\}\end{matrix}\right\} & \ddots & \emptyset \\ \vdots & \vdots & \ddots & \vdots \\ \left\{\begin{matrix}\{y_{(1)}, d_{r,1}\}\\ \vdots \\ \{y_{(m)}, d_{r,1}\}\end{matrix}\right\} & \left\{\begin{matrix}\{y_{(1)}, d_{r,2}\}\\ \vdots \\ \{y_{(m)}, d_{r,2}\}\end{matrix}\right\} & \cdots & \emptyset \end{bmatrix} \tag{27}$$

Proof. Proposition 4.5 indicates D_{Agg}, which extends further to $D_{\widetilde{Agg}}$ in proposition 4.6. Proposition 6.1 develops the syntactic form and semantic form of the collection of compound fuzzy sets $\{d_{ij} : i = 1, \ldots, r; j = 1, \ldots, p\}$ for the decision attitude. The D_{Agg} can be applied to $f_{\overline{X}}(HD)$. Thus, the form of $D"_{\widetilde{Agg}}$ is derived. □

Regarding the final selection of the representation of 2^{nd} degree DA fuzzy sets and AOs, two definitions are formed.

Definition 6.3. If an aggregation operator has more than one of the 2^{nd} degree DA fuzzy sets, the selection of DAs for the dedicated AO is of the form:

$$d"^{*}(k) = ArgMax\left(\left\{\left\{y_{(k)}, d_{i,j}\right\} : d_{i,j} \neq \emptyset\right\}\right) \tag{28}$$

$d"^{*}(k)$ returns the index of the linguistic label to describe the AO.

Definition 6.4. If the 2^{nd} degree DA fuzzy set d_{ij} includes more than one aggregation operator, the selection of AOs of d_{ij} is of the form:

$$d^{*}_{ij} = ArgMax\left(\left\{\left\{y_{(1)}, d_{ij}\right\}, \ldots, \left\{y_{(k)}, d_{ij}\right\}, \ldots, \left\{y_{(m)}, d_{ij}\right\}\right\}\right) \tag{29}$$

d^{*}_{ij} returns the index in $\vec{Y}$ to represent the linguistic label d_{ij}.

Algorithm 6.5. $DAAO\text{-}2 = CSAO\text{-}2\left(D, \widetilde{Agg}, X, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$ **:**
Input:

a. A collection of the 1^{st} degree DA linguistic variable: $D = \{d_1, \cdots, d_j, \cdots, d_p\}$ is comprised of the membership set $\{\mu_{d^j}\}$ and the corresponding inverse membership set $\left\{\mu^{-1}_{d^j, \varphi = '-', '\theta', '+'}\right\}$ with the tuning factor set $\{\tau_{d^j}\}$;
b. A vector of hedge terms $\overrightarrow{V_h}$ and A vector of directional terms $\overrightarrow{V_d}$;
c. A collection of AOs: $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_m)$;
d. A collection of information granules: $X = (x_1, \ldots, x_i, \ldots, x_n)$;
e. A collection of the parameters of the member fuzziness process: $\left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)$;

Process:

Step 1: Compute $\widetilde{Agg}(X)$, and then $Y = (y_1, \ldots, y_k, \ldots, y_m)$ is achieved;
Step 2: Get the permutation of Y: $\vec{Y} = \left\{y_{(1)}, \ldots, y_{(k)}, \cdots, y_{(m)}\right\}$;
Step 3: Get $[y_*, y^*] = \left[y_{(1)}, y_{(m)}\right]$;
Step 4: Calculate intervals and $\{(\gamma_{d^j}, \Delta_{d^j})\}_{j=1}^{p}$ for $D$by equally diving $[y_*, y^*]$;

i. $d_1 = \left(y_*, y_*, y_* + \frac{y^* - y_*}{p-1}\right) = (\gamma_{d^1}, \gamma_{d^1}, \gamma_{d^1} + \Delta_{d^1})$
ii. $d_{j \neq 1,p} = \left(y_* + \frac{y^* - y_*}{p-1}(j-2), y_* + \frac{y^* - y_*}{p-1}(j-1), y_* + \frac{y^* - y_*}{p-1}(j)\right)$

$$= (\gamma_{d^j} - \Delta_{d^j}, \gamma_{d^j}, \gamma_{d^j} + \Delta_{d^j})$$

iii. $d_p = \left(y^* - \frac{y^* - y_*}{p-1}, y^*, y^*\right) = (\gamma_{d^p} - \Delta_{d^p}, \gamma_{d^p}, \gamma_{d^p})$

Step 5: Elicit memberships μ_{d^j} for $D$by interpolation of (a,b,c).
Step 6: Calculate $D\left(\vec{Y}\right)$, $D_{\widetilde{Agg}}$ and $d^{*}(k)$, $\forall k$.
Step 7: Form HD with rating interface by Algorithm 5.1.
Step 8: Calculate $\{d_{ij} : i = 1, \ldots, r; j = 1, \ldots, p\}$ of HD by
$f_{\overline{X}}\left(\left\{\left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \mu^{-1}_{d^j\varphi}\right)\right\}, [y_*, y^*], \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$ (Algorithm 5.2)
Step 9: Calculate $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$
Step 10: Calculate $d"^{*}(k)$, $\forall k$ in $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$.
Step 11: Calculate d^{*}_{ij}, $i = 1, \ldots, r; j = 1, \ldots, p$

Output: $\left\{d^{*}_{ij}\right\}$ **//END**

Example 6.6. This example is a continuation of Example 4.12. DAAO-2 is illustrated as follows.

Input:

a. $D = \{d_1, d_2, d_3\} = \{\text{P,N,O}\}$;
μ_{d^j} is the symmetric triangular membership, $\forall j \in \{1,2,3\}$;
$\mu^{-1}_{d^j, \varphi='-','\theta','+'}$ is the inversed triangular membership set, $\forall j \in \{1,2,3\}$;
$\tau_{d^j} = 1, \quad \forall j \in \{1,2,3\}$;

b. $\overrightarrow{V_h} = [\text{Little,Quite,Much}]$, and $\overrightarrow{V_d} = [\text{Below,Absolutely,Above}]$

c. A collection of AOs: $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_{17})$;

d. A collection of information granules:
$C = (0.4, 0.5, 0.6, 0.7, 0.9)$,
$W = owaW\,(0.6, 5) = (0.3801, 0.1964, 0.1589, 0.1387, 0.1253)$,
and thus $X = ((0.4, 0.1978), \ldots, (0.9, 0.6250))$;

e. A collection of the parameters of the member fuzziness process:
$\left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right) = (\{1,2,3\}, 0.5)$;

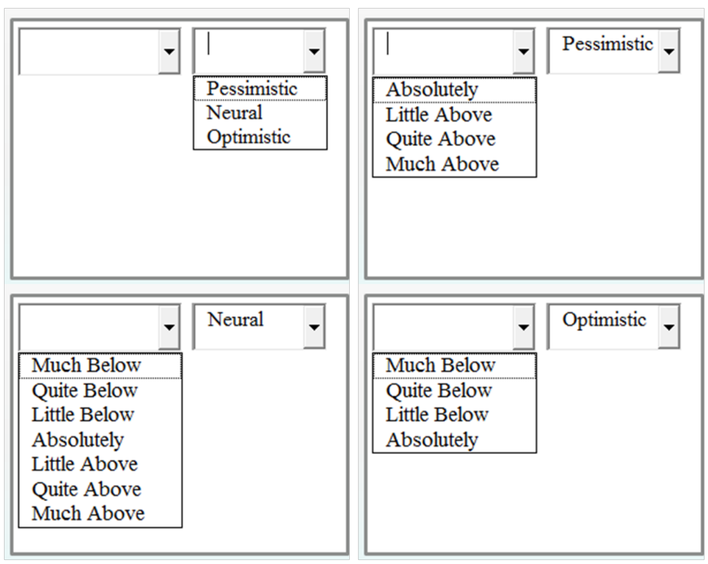

FIGURE 4. Deductive Rating Strategy in the Rating Interface of CLOS

Process:

Step 1-3: $Y = \widetilde{Agg}\,(X) = \left\{ \begin{array}{l} 0.5375, 0.5137, 0.5332, 0.5557, 0.6949, 0.3807, 0.4, 0.6939, \\ 0.5019, 0.4619, 0.5127, 0.5, 0.5, 0.5, 0.1193, 0.5199, 0.4868 \end{array} \right\}$;

$\vec{Y} = \left\{ \begin{array}{l} 0.1193, 0.3807, 0.4, 0.4619, 0.4868, 0.5, 0.5, 0.5, 0.5019, 0.5127, \\ 0.5137, 0.5199, 0.5332, 0.5375, 0.5557, 0.6939, 0.6949 \end{array} \right\}$;

$[y_*, y*] = \left[y_{(1)}, y_{(m)}\right] = [0.1193, 0.6949]$;

Step 4: Calculate intervals and $\{(\gamma_{d^j}, \Delta_{d^j})\}_{j=1}^{3}$ for D:

i. $d_1 = (0.1193, 0.1193, 0.4071)$;
ii. $d_2 = (0.1193, 0.4071, 0.6949)$;
iii. $d_3 = (0.4071, 0.6949, 0.6949)$;
iv. $\{\gamma_{d^j}\} = \{0.1193, 0.4071, 0.6949\}$ and $\Delta_{d^j} = 1, \forall j \in \{1,2,3\}$;

Step 5: Elicit memberships μ_{d^j} for D. The results are shown in Figure 3.

Step 6: Calculate $D\left(\vec{Y}\right) = \left\{D\left(y_{(k)}\right)\right\}$, $D_{\widetilde{Agg}}$ and $d^*(k)$, $\forall k$. The results are shown in Table 3.

Step 7: Form HD with rating interface by Algorithm 5.1.

$\overrightarrow{V_{hd}} = [v_{hd_1}, \cdots, v_{hd_7}]$= [Much Below,Quite Below, Little Below, Absolutely, Little Above,Quite Above,Much Above], thus

$$HD = G_{\aleph}(H, D) = \begin{bmatrix} \emptyset & MB-N & MB-O \\ \emptyset & QB-N & QB-O \\ \emptyset & LB-N & LB-O \\ A-P & A-N & A-O \\ LA-P & LA-N & \emptyset \\ QA-P & QA-N & \emptyset \\ MA-P & MA-N & \emptyset \end{bmatrix}$$

And the rating interface is shown in Figure 4.

Step 8: Calculate $\{d_{ij}\}$ of HD by

$f_{\overline{X}}\left(\left\{\left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \mu^{-1}_{d^{j\varphi}}\right)\right\}, [y_*, y^*], \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$ (Algorithm 5.2). Thus,

$$\{d_{ij}\} = \begin{pmatrix} \emptyset & (0.1193, 0.2092, 0.2992) & (0.4071, 0.4971, 0.5870) \\ \emptyset & (0.2392, 0.3112, 0.3831) & (0.5270, 0.5990, 0.6709) \\ \emptyset & (0.3472, 0.3771, 0.4071) & (0.6350, 0.6649, 0.6949) \\ (0.1193, 0.1193, 0.1193) & (0.4071, 0.4071, 0.4071) & (0.6949, 0.6949, 0.6949) \\ (0.1193, 0.1493, 0.1793) & (0.4071, 0.4371, 0.4671) & \emptyset \\ (0.1433, 0.2152, 0.2872) & (0.4310, 0.5030, 0.5750) & \emptyset \\ (0.2272, 0.3172, 0.4071) & (0.5150, 0.6050, 0.6950) & \emptyset \end{pmatrix}$$

Step 9: Calculate $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$

$$\begin{pmatrix} \emptyset & (0^{17}) & \begin{pmatrix} 0^3, 0.6095, 0.8860, 0.9672, \\ 0.9672, 0.9672, 0.9463, \\ 0.8265, 0.8152, 0.7455, \\ 0.5978, 0.5503, 0.3482, 0^2 \end{pmatrix} \\ \emptyset & (0^1, 0.0332, 0^{16}) & \begin{pmatrix} 0^{12}, 0.0860, 0.1455, \\ 0.3981, 0^2 \end{pmatrix} \\ \emptyset & (0^1, 0.8799, 0.2372, 0^{14}) & (0^{17}) \\ (1, 0^{16}) & (0^{17}) & (0^{16}, 1, 1) \\ (0^{17}) & (0^3, 0.1716, 0^{13}) & \emptyset \\ (0^{17}) & \begin{pmatrix} 0^3, 0.4285, 0.7742, 0.9576, \\ 0.9576, 0.9576, 0.9838, \\ 0.8665, 0.8523, 0.7652, \\ 0.5806, 0.5212, 0.2686, 0^2 \end{pmatrix} & \emptyset \\ \begin{pmatrix} 0^1, 0.2933, \\ 0.0791, 0^{14} \end{pmatrix} & \begin{pmatrix} 0^{11}, 0.0545, 0.2022, 0.2497, \\ 0.4518, 0^2 \end{pmatrix} & \emptyset \end{pmatrix}$$

"0" means that the membership of AO is equal to zero in this compound linguistic term. The index of "0" means the number of zeros.

Step 10: Calculate $d"^*(k)$, $\forall k$ in $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$. The results are shown in Table 5.

(k)	Agg	$y_{(k)}$	$D"\left(y_{(k)}\right)$	$d"^*(k)$
1	$wmed_f$	0.1193	{A-P(1)}	A-P
2	$ow\max$	0.3807	{MA-P(0.293),QB-N(0.033),LB-N(0.880)}	LB-N
3	$ow\min$	0.4	{MA-P(0.079),QB-N(0.237)}	QB-N
4	$wmed$	0.4619	{LA-N(0.172),QA-N(0.428),MB-P(0.609)}	MB-P
5	$wmed_{ss}$	0.4868	{QA-N(0.774),MB-O(0.886)}	MB-O
6	$wmed_{mm}$	0.5	{QA-N(0.958),MB-O(0.967)}	MB-O
7	$wmed_{dp}$	0.5	{QA-N(0.958),MB-O(0.967)}	MB-O
8	$wmed_y$	0.5	{QA-N(0.958),MB-O(0.967)}	MB-O
9	wgo	0.5019	{QA-N(0.984),MB-O(0.946)}	QA-N
10	$wmed_l$	0.5127	{QA-N(0.866),MB-O(0.827)}	QA-N
11	whm	0.5137	{QA-N(0.852),MB-O(0.815)}	QA-N
12	$wmed_{ws}$	0.5199	{QA-N(0.765),MA-N(0.054),MB-O(0.745)}	QA-N
13	wgm	0.5332	{QA-N(0.581),MA-N(0.202),MB-O(0.598),QB-O(0.086)}	MB-O
14	wrp	0.5375	{QA-N(0.521),MA-N(0.250),MB-O(0.550),QB-O(0.145)}	MB-O
15	wam	0.5557	{QA-N(0.269),MA-N(0.452),MB-O(0.348),QB-O(0.398)}	MA-N
16	owa	0.6949	{A-O(1)}	A-O
17	Leximin	0.6949	{A-O(1)}	A-O

TABLE 5. The Results for $D"\left(\vec{Y}\right)$ and $d"^*(k)$ of 17 AOs

Step 11 and Return: Calculate d^*_{ij}, $i = 1, \ldots, r; j = 1, \ldots, p$

$$\left\{d^*_{ij}\right\} = d^*_{ij}\left(\begin{bmatrix} \emptyset & MB-N & MB-O \\ \emptyset & QB-N & QB-O \\ \emptyset & LB-N & LB-O \\ A-P & A-N & A-O \\ LA-P & LA-N & \emptyset \\ QA-P & QA-N & \emptyset \\ MA-P & MA-N & \emptyset \end{bmatrix}\right) = \begin{bmatrix} \emptyset & 0 & 6,7,8 \\ \emptyset & 2 & 15 \\ \emptyset & 2 & 0 \\ 1 & 0 & 16,17 \\ 0 & 4 & \emptyset \\ 0 & 9 & \emptyset \\ 2 & 15 & \emptyset \end{bmatrix}$$

"0" means no AO is available in this compound linguistic term. Another number means the index in $\vec{Y}$.

If a linguistic term (e.g. MB-O, A-O) includes more than one AOs (e.g. (6,7,8) or (16,17)), either of the AOs can be used since the AOs produce the same result with respect to a compound fuzzy set.

Example 6.7. Using DAAO-2, this example considers only seven AOs used in Example 4.13. Steps 1 and 7 are skipped. The remains of the steps are illustrated as follows:

Step 8**:** Calculate $\{d_{ij}\}$

$$\{d_{ij}\} = \begin{pmatrix} \emptyset & (0.4619, 0.4983, 0.5347) & (0.5784, 0.6148, 0.6512) \\ \emptyset & (0.5105, 0.5396, 0.5687) & (0.6270, 0.6561, 0.6852) \\ \emptyset & (0.5542, 0.5663, 0.5784) & (0.6707, 0.6828, 0.6949) \\ (0.4619, 0.4619, 0.4619) & (0.5784, 0.5784, 0.5784) & (0.6949, 0.6949, 0.6949) \\ (0.4619, 0.4741, 0.4862) & (0.5784, 0.5906, 0.6027) & \emptyset \\ (0.4716, 0.5008, 0.5299) & (0.5881, 0.6173, 0.6464) & \emptyset \\ (0.5056, 0.5402, 0.5784) & (0.6221, 0.6585, 0.6949) & \emptyset \end{pmatrix}$$

Step 9**:** Calculate $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$.

$$D"_{\widetilde{Agg}}\left(\vec{Y}\right) = \begin{pmatrix} \emptyset & \begin{pmatrix} 0^1, 0.9025, 0.5786, \\ 0.0415, 0^3 \end{pmatrix} & (0^7) \\ \emptyset & \begin{pmatrix} 0^2, 0.1101, 0.7813, \\ 0.9282, 0.4477, 0^1 \end{pmatrix} & (0^7) \\ \emptyset & (0^5, 0.1255, 0^1) & (0^7) \\ (1, 0^6) & (0^7) & (0^7, 1) \\ (0^7) & (0^7) & \emptyset \\ (0^2, 0.9615, 0.5566, 0^4) & (0^7) & \emptyset \\ \begin{pmatrix} 0^2, 0.2214, 0.7584, \\ 0.8759, 0.6248, 0^1 \end{pmatrix} & (0^7) & \emptyset \end{pmatrix}$$

“0” means that the membership of AO is equal to zero in this compound linguistic term. The index of “0” means the number of zeros.

Step 10**:** Calculate $d"^*(k)$, $\forall k$ in $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$, which are shown in Table 6.

Step 11**:** and Return : $\left\{d^*_{ij}\right\}$ is shown as follows.

$$\left\{d^*_{ij}\right\} = d^*_{ij}\left(\begin{bmatrix} \emptyset & MB-N & MB-O \\ \emptyset & QB-N & QB-O \\ \emptyset & LB-N & LB-O \\ A-P & A-N & A-O \\ LA-P & LA-N & \emptyset \\ QA-P & QA-N & \emptyset \\ MA-P & MA-N & \emptyset \end{bmatrix}\right) = \begin{bmatrix} \emptyset & 2 & 0 \\ \emptyset & 5 & 0 \\ \emptyset & 6 & 0 \\ 1 & 0 & 7 \\ 0 & 0 & \emptyset \\ 2 & 0 & \emptyset \\ 5 & 0 & \emptyset \end{bmatrix}$$

“0” means no AO is available in this compound linguistic term. Another number means the index in $\vec{Y}$.

One can purely use DAAO-1, or DAAO-2. However, the selection function by $ArgMax$ is excessively straightforward in DAAO-1 in many AO candidates for one DA linguistic term d_j , whilst DAAO-2 contains no AOs for some linguistic terms if insufficient AO candidates for the relatively large scale of the compound linguistic terms. Regarding the number of AO candidates, the selection strategy to combine DAAO-1 and DAAO-2 is of the following algorithm.

(k)	Agg	$y_{(k)}$	$D"\left(y_{(k)}\right)$	$d"^{*}(k)$
1	$wmed$	0.4619	{A-P(1)}	A-P
2	wgo	0.5019	{QA-P(0.962),MB-N(0.903)}	QA-P
3	whm	0.5137	{QA-P(0.557),MA-P(0.221),MB-N(0.579),QB-N(0.110)}	MB-N
4	wgm	0.5332	{MA-P(0.758),MB-N(0.042),QB-N(0.781)}	QB-N
5	wrp	0.5375	{MA-P(0.876),QB-N(0.928)}	QB-N
6	wam	0.5557	{MA-P(0.625),QB-N(0.448),LB-N(0.126)}	MA-P
7	owa	0.6949	{A-O(1)}	A-O

TABLE 6. The Results for $D"\left(\vec{Y}\right)$ and $d"^{*}(k)$ of Seven AOs

Algorithm 6.8. (Selection Strategy, $SAO\left((d_j, h_i), \left(D_{\widetilde{Agg}}, d"^{*}(k)\right)\right)$):
Input: $D_{\widetilde{Agg}}$ of DAAO-1, and $d"^{*}(k)$ of DAAO-2.

Selection Process:

Step 1: Select an atomic term of DA d_j.

Step 2: Check if no AO return for the d_j in $D_{\widetilde{Agg}}$,
True: Return empty message and go to Step 1.
False: Go to Step 3.

Step 3: Check if only one AO return for the d_j in $D_{\widetilde{Agg}}$,
True: Return $Agg_{(k)}$.
False: Go to Step 4.

Step 4: Select the directional hedge term h_i.

Step 5: Check if no AO return for the $d_{ij} = h_i \oplus d_j$ in $d"^{*}(k)$,
True: Return empty message and go to Step 4 or 1.
False: Return $Agg_{(k)} = d"^{*}(k)$.

Return: $Agg_{(k)}$. **//End**

Example 6.9. Consider Examples 4.13 and 6.7. The rating interface can be referred to Figure 4. Three cases are illustrated.

Case 1: d_3= 'Opt'.
Input: $D_{\widetilde{Agg}}$ of DAAO-1 in table 4 and $d"^{*}(k)$ of DAAO-2 in Table 6.

Selection Process

Step 1: Select an atomic term of DA: d_3="Opt".
Step 2: owa return for the d_j in $D_{\widetilde{Agg}}$,
Step 3: Only one AO return for the d_j in $D_{\widetilde{Agg}}$,
Return: $Agg_{(7)} = owa$

Case 2: d_2="Ntl".
Input: $D_{\widetilde{Agg}}$ of DAAO-1 in table 4 and $d"^{*}(k)$ of DAAO-2 in Table 6.

Selection Process:

Step 1**:** Select an atomic term of DA: d_2="Ntl".

Step 2 **and** 3**:** wgm, wrp, and wam return for the d_j.

Step 4**:** Select the directional hedge term h_i.

Step 5**:** Check if no AO return for the $d_{ij} = h_i \oplus d_j$ in $d"^* (k)$,

True: Return empty message and go to Step 4 (As d_j="Ntl" is assumed,
Step 1 is skipped).
False: Return $Agg_{(k)} = d"^* (k)$.

Return: $Agg_{(k)} = wgo, wrp, wam$ depends on which valid h_i is firstly selected.

Case 3**:** which d_1="Pes", is similar to Case 2. $Agg_{(k)} = wmed, wgo, wrp$ depends on which valid h_i is firstly selected.

7. CSAO in Decision Matrix

In a decision matrix, more than one alternative is considered. This means different input value sets$X$'s possibly produce different $\{d^* (k)\}$, $\left\{d_j^*\right\}$, and $\left\{d_{ij}^*\right\}$. To address this issue, three definitions are created as follows:

Definition 7.1. In the decision matrix, the linguistic presentation of the style of the decision attitude for the AOs is computed by the form:

$$\left\{d_\beta^* (k)\right\}^* = Max \left(Mode \left(Join \left(\left\{d_\beta^* (k)\right\}\right)\right)\right), \tag{30}$$

where β is the index of the alternative of the decision matrix. $Join$ is the function which combines the matrices, and $Mode$ is the value that occurs the most frequently in an entry of $Join \left(\left\{\left\{d_{ij}^*\right\}_\beta\right\}\right)$.

Definition 7.2. In a decision matrix, the AO of the style of the decision attitude for the linguistic terms is computed as:

$$\left\{\left\{d_j^*\right\}_\beta\right\}^* = Max \left(Mode \left(Join \left(\left\{\left\{d_j^*\right\}_\beta\right\}\right)\right)\right) \tag{31}$$

Definition 7.3. let $\left\{d_{ij}^*\right\}_\beta$ be the DAAO-2 pattern of the alternativeβ. Then, the pattern of the decision matrix is of the form:

$$\left\{\left\{d_{ij}^*\right\}_\beta\right\}^* = Max \left(Mode \left(Join \left(\left\{\left\{d_{ij}^*\right\}_\beta\right\}\right)\right)\right) \tag{32}$$

If more than one AO index is returned in the entry, the index number with the highest value is chosen since it is likely to produce higher value for each alternative of the decision matrix. Thus the Max is taken. Also Max can eliminate "0" values. The Selection Strategy in Decision matrix is illustrated in Algorithm 7.4.

Algorithm 7.4. $(Agg_{(k)} = \overline{CSAO} \left((h_i, d_j), \{X\}, \widetilde{Agg}, D, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$:

Input: $(h_i, d_j), D, \widetilde{Agg}, X, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)$

Process:

Step 1: Calculated $d_\beta^*(k)$ in $CSAO1\left(D, \widetilde{Agg}, X_\beta\right) \quad \forall\beta \in \{1,\ldots,|\{X\}|\}$ (Algorithm 4.10)

Step 2: $\{d_j^*\}_\beta = CSAO1\left(D, \widetilde{Agg}, X_\beta\right)$, $\forall\beta \in \{1,\ldots,|\{X\}|\}$ (Algorithm 4.10)

Step 3: $\{d_{ij}^*\}_\beta = CSAO2\left(D, \widetilde{Agg}, X_\beta, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$, $\forall\beta \in \{1,\ldots,|\{X\}|\}$ (Algorithm 5.1)

Step 4: $\left\{d_\beta^*(k)\right\}^* = Max\left(Mode\left(Join\left(\left\{d_\beta^*(k)\right\}\right)\right)\right)$

Step 5: $\left\{\{d_j^*\}_\beta\right\}^* = Max\left(Mode\left(Join\left(\left\{\{d_j^*\}_\beta\right\}\right)\right)\right)$

Step 6: $\left\{\{d_{ij}^*\}_\beta\right\}^* = Max\left(Mode\left(Join\left(\left\{\{d_{ij}^*\}_\beta\right\}\right)\right)\right)$

Step 7: Check if no AO return for the d_j in$\left\{d_\beta^*(k)\right\}^*$,

True: Return Empty message and go to Input to request another d_j.
False: Go to Step 4.

Step 8: Check the numbers of AO's return for the d_j in $\left\{d_\beta^*(k)\right\}^*$,

1: Return$Agg_{(k)} = \left\{d_\beta^*(k)\right\}^*$ without considering h_i.
2-3: Return $Agg_{(k)} = \left\{\{d_j^*\}_\beta\right\}^*$ without consideringh_i.
≥ 4 : Go to Step 9.

Step 9: Check if no AO return for the $d_{ij} = h_i \oplus d_j$ in $\left\{\{d_{ij}^*\}_\beta\right\}^*$,

True: Return empty message and go to **Input** with new (h_i, d_j).
False: Return $Agg_{(k)} = \left\{\{d_{ij}^*\}_\beta\right\}^*$.

Return: $Agg_{(k)}$. **//End**

The use of this algorithm is shown in the section 7.3. The next section performs the numerical analyses for the proposed DSAO model to validate its usability and validity.

8. **Numerical Analyses**

Three major analyses are performed and discussed as follows.

8.1. **Scenario.** Consider a decision matrix as follows,

$$\bar{O} = \begin{matrix} W \\ C \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{matrix} \begin{matrix} (w_1 & w_2 & w_3 & w_4 & w_5) \\ (c_1 & c_2 & c_3 & c_4 & c_5) \\ \end{matrix} \begin{pmatrix} 0.5 & 0.5 & 0.6 & 0.7 & 0.9 \\ 0.5 & 0.7 & 0.9 & 0.8 & 0.5 \\ 0.6 & 0.9 & 0.5 & 0.7 & 0.5 \\ 0.4 & 0.5 & 0.6 & 0.8 & 0.9 \\ 0.5 & 0.9 & 0.5 & 0.7 & 0.5 \end{pmatrix},$$

where $W = owaW(\delta), \delta \in \{0.1, 0.2, \ldots, 1\}$, which is shown in Table 7.

In this section, firstly, ten different decision matrices of the above form are created with 10 weight sets (Table 7). The matrices are further aggregated by 10 aggregation operators defined as: $\widetilde{Agg} = (whm, wgm, wam, wmed, wrp01, wrp05, wrp20, wgo01, wgo05, wgo09)$, where 01 means $\alpha = 0.1$, and so on.

Secondly, regarding discussion of the research values, the decision matrix with $\alpha = 0.9$ is selected for the application of DSAO-2.

δ	w_1	w_2	w_3	w_4	w_5
0.1	0.851	0.061	0.038	0.028	0.022
0.2	0.725	0.108	0.070	0.053	0.044
0.3	0.617	0.143	0.098	0.077	0.065
0.4	0.525	0.168	0.122	0.099	0.085
0.5	0.447	0.185	0.142	0.120	0.106
0.6	0.381	0.196	0.159	0.139	0.125
0.7	0.324	0.202	0.173	0.156	0.145
0.8	0.276	0.205	0.184	0.172	0.163
0.9	0.235	0.203	0.193	0.187	0.182
1	0.200	0.200	0.200	0.200	0.200

TABLE 7. W Generated by $owaW(\delta), \delta \in \{0.1, 0.2, \ldots, 1\}$

8.2. **Properties of Individual AOs.** Ten decision matrices of the variation of weight sets are aggregated by ten AOs. The weight sets are generated by $owaW(\delta), \delta \in \{0.1, 0.2, \ldots, 1\}$ and are shown in Table 7. The larger δ means the less gap among the individual weights. When $\delta = 1$, all weights are of equal values. The data are plotted in Figures 5 and 6 .

Figures 5 and 6 show that different AOs behave differently for different decision matrices. This means that each AO has a different style. *wrp and wgo* with different α produce different results and likely different ranks. This means that a different AO with different α can have its own style.

Although $w_1 > w_2 > \ldots > w_5$ except for $\delta = 1$, the distribution among the weights are narrowed whilst δ increases. The sensitivity of each AO for the changes of weight is different. When the difference among the weights get less (e.g. increase of δ), the outputs of *wgo05 and wgo09* decrease while the outputs of others AOs increase. In addition, *wmed* has relative sensitivity of the change of the values of weights.

Regarding the patterns of the AO population in the figures, the figures show that the lines of AOs are closer while δ decreases. When δ increases, which means the gap of the weights of the criteria is reduced, the lines get farther apart. The main reason is that the criteria in a high index becomes more significant, and the values of the criteria in a higher index are more than the values of the criteria in a lower index.

Regarding the patterns of CSAO, the number of the AOs in Opt should be more than the number of the AOs in Pes. The main reason is that more lines are located

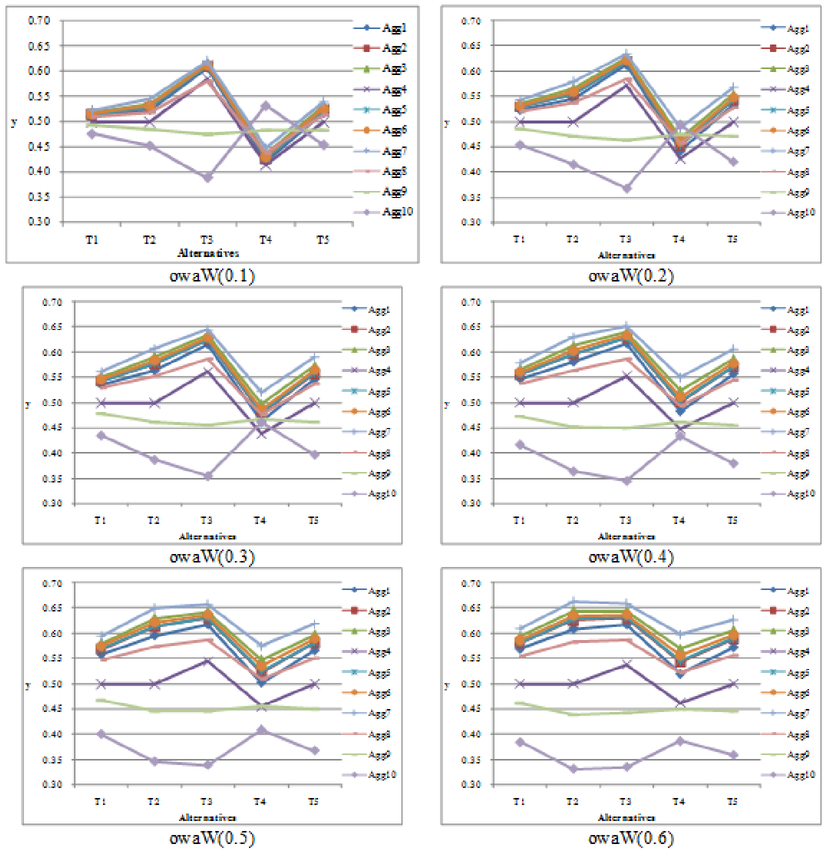

FIGURE 5. Results of Individual Aggregation Operators (Part I)

in upper position of the y-axis. In the next sub-section this issue is investigated in depth.

8.3. **Selection of AO by CSAO.** What a decision maker finally feels of interest is not the properties of the aggregation operators, but which AO is the most suitable. In fact, there is likely no absolute answer. In the real world, no decision maker can always guarantee an absolutely accurate answer (except for those who are arrogant), but the best and the most appropriate answer which he think it is correct (but others may not agree). Similarly, why they make different decisions when the objective situation and background are the same? One of the explanations is that they have different cognitive styles or individual differences. Some make clever decisions whilst some do not. In the mathematician's view, how they make decision can be modeled by equations. In the CSAO model, each AO reflects a different cognitive style. CSAO is used to classify the cognitive styles. This research proposes that

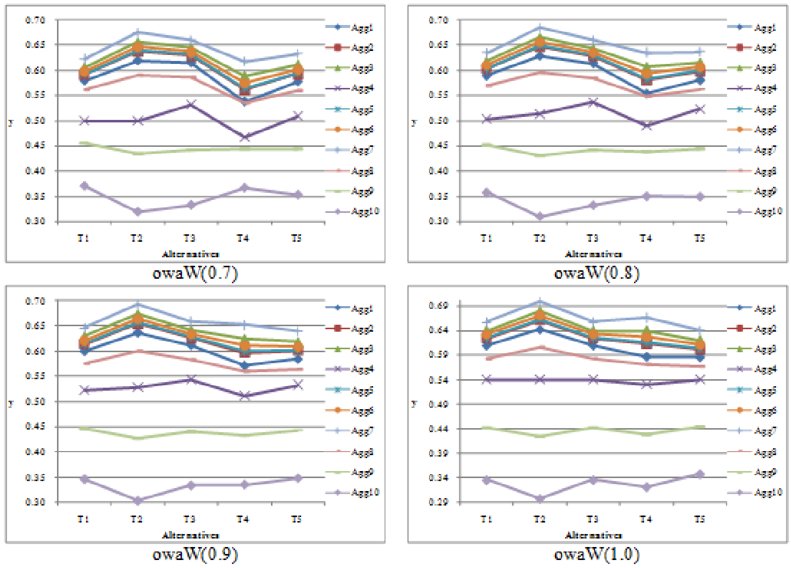

FIGURE 6. Results of Individual Aggregation Operators (Part II)

CSAO is represented by DAAO-1 and DAAO-2. The rating interface can be referred to Figure 4.

Tables 8 and 9 show the $\left\{d_{\beta}^{*}(k)\right\}^{*}$ and $\left\{\left\{d_{j}^{*}\right\}_{\beta}\right\}^{*}$ of DAAO-1 of the proposed decision matrix where $W = owaW(0.9)$. Interestingly, no matter which alternative input set of the decision matrix is used, the order of the AOs (k) is preserved to be the same.

(k)	Agg	$d_1^*(k)$	$d_2^*(k)$	$d_3^*(k)$	$d_4^*(k)$	$d_5^*(k)$	$\left\{d_{\beta}^{*}(k)\right\}^{*}$
1	$wrp01$	Pes	Pes	Pes	Pes	Pes	Pes
2	$wrp05$	Ntl	Ntl	Ntl	Ntl	Ntl	Ntl
3	$wgo05$	Ntl	Ntl	Ntl	Ntl	Ntl	Ntl
4	wam	Opt	Opt	Opt	Ntl	Ntl	Opt
5	$wrp20$	Opt	Opt	Opt	Ntl	Opt	Opt
6	$wgo01$	Opt	Opt	Opt	Opt	Opt	Opt
7	$wgo09$	Opt	Opt	Opt	Opt	Opt	Opt
8	$wmed$	Opt	Opt	Opt	Opt	Opt	Opt
9	wgm	Opt	Opt	Opt	Opt	Opt	Opt
10	whm	Opt	Opt	Opt	Opt	Opt	Opt

TABLE 8. The Linguistic Presentation of the Style of the Decision Attitude for the AOs of the Decision Matrix $\{d^*(k)\}^*$

j	d	$\{d_j^*\}_1$	$\{d_j^*\}_2$	$\{d_j^*\}_3$	$\{d_j^*\}_4$	$\{d_j^*\}_5$	$\left\{\{d_j^*\}_\beta\right\}^*$
1	Pes	1	1	1	1	1	1
2	Ntl	3	3	3	3	3	3
3	Opt	10	10	10	10	10	10

TABLE 9. The AO of the Style of the Decision Attitude for the Linguistic Terms of the Decision Matrix $\left\{\{d_j^*\}_\beta\right\}^*$

If the decision maker chooses *Opt*, there are seven options to represent the Optimistic AO. It is too subjective to use ArgMax in equation 21, thus DAAO-2 is needed. From DAAO-2 (Algorithm 5.1), $\{d_{ij}^*\}_1$, $\{d_{ij}^*\}_2$,$\{d_{ij}^*\}_3$,$\{d_{ij}^*\}_4$,$\{d_{ij}^*\}_5$ are as below respectively:

$$\begin{bmatrix} \emptyset & 0 & 3 \\ \emptyset & 2 & 5 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \emptyset \\ 0 & 3 & \emptyset \\ 2 & 5 & \emptyset \end{bmatrix}, \begin{bmatrix} \emptyset & 0 & 3 \\ \emptyset & 2 & 5 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \emptyset \\ 0 & 3 & \emptyset \\ 2 & 5 & \emptyset \end{bmatrix}, \begin{bmatrix} \emptyset & 0 & 3 \\ \emptyset & 2 & 5 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 0 & \emptyset \\ 0 & 3 & \emptyset \\ 2 & 5 & \emptyset \end{bmatrix}, \begin{bmatrix} \emptyset & 2 & 4 \\ \emptyset & 2 & 7 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \emptyset \\ 0 & 4 & \emptyset \\ 2 & 7 & \emptyset \end{bmatrix}, \begin{bmatrix} \emptyset & 0 & 3 \\ \emptyset & 2 & 5 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 0 & \emptyset \\ 0 & 3 & \emptyset \\ 2 & 6 & \emptyset \end{bmatrix}$$

From equation 32 in Definition 7.3, then

$$\{d_{ij}^*\}^* = Max\left(Mode\left(Join\left(\left\{\{d_{ij}^*\}_\beta\right\}\right)\right)\right) = \begin{bmatrix} \emptyset & 0 & 3 \\ \emptyset & 2 & 5 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \emptyset \\ 0 & 3 & \emptyset \\ 2 & 5 & \emptyset \end{bmatrix}$$

If a decision maker chooses “Pes” for the AO in the decision system, in the first rating step, there is only one choice, $wrp01$, as it is indicated in Table 8. The second rating category in V_{hd} is unnecessary.

If “Ntl” is chosen, for the representation of AO, $wrp05$ and $wgo05$ are the candidates, by using equation 7, where $wgo05$ is for “Ntl”.

When “Opt” is chosen, there are seven candidates. It is too straightforward to use ArgMax (equation 21). Thus the second rating category in V_{hd} is needed. The index of the AO can be found in $\{d_{ij}^*\}^*$. $wgo05$,$wrp20$,$wgo09$ and wgm are the options with respect to the choice of the second rating linguistic term.

9. Conclusions

As different aggregation operators produce different results, these results can be described by the possibility likelihoods of the cognitive styles. The selection of the aggregation operators is related to the likelihoods of the cognitive styles of the operators. To achieve the proposal, the Cognitive Style and Aggregation Operator (CSAO) model is proposed to analyze the mapping relationship between aggregation operators and cognitive style on the basis of fuzzy set theory. The CSAO model has two types of Decision Attitude and Aggregation Operator (DAAO) model: DAAO-1, DAAO-2. The difference is that DAAO-1 applies classical single dimension linguistic terms whilst DAAO-2 applies the compound linguistic terms. Three Algorithms for AO selection are developed.

The appropriate operators will be chosen according to the linguistic terms of the decision attitudes in the CSAO model. The cognitive style is characterized by the decision attitude. The CSAO model is useful for measuring the distribution of the AOs.

Examples 4.12 and 6.6 test 17 AOs. On the basis of the result pattern, Examples 4.13 and 6.7 select only 7 AOs. From the numerical examples, it can be concluded that the weighted median with other t-connorms and t-norms, *owmax*, *owmin*, and *owa* is not appropriate for the aggregation of the decision matrix. The reasons are stated after the numerical Examples 4.12 and 4.13.

In the section of numerical analyses, 10 AOs are tested for 10 decision matrices. The best practices of AO selection are illustrated using the combination of DAAO-1 and DAAO-2.

Limitation of the CSAO model is that the CSAO relies on the definitions of the candidates. If some candidates are abnormal, the CSAO pattern will be abnormal too. Usually the abnormal operators produce excessively optimistic or excessively positive results. In this case, the expert can remove the abnormal AO by his perception, and then recalculate the patterns again. After several refinements of the patterns, the appropriate CSAO model can be developed.

The CSAO is devoted to a proposal as how to map a collection of aggregation operators into a collection of decision attitudes by the CSAO model. This model is typically useful for those unsolved issues in the selection of aggregation operators. The OA candidates are determined by the decision maker with respect to the cognitive styles, which are characterized by decision attitudes. Thus the CSAO model is useful for the decision making applications with consideration of the cognitive styles (or decision attitudes) of the decision makers.

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FUZZY GOAL PROGRAMMING TECHNIQUE TO SOLVE MULTIOBJECTIVE TRANSPORTATION PROBLEMS WITH SOME NON-LINEAR MEMBERSHIP FUNCTIONS

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Abstract. The linear multiobjective transportation problem is a special type of vector minimum problem in which constraints are all equality type and the objectives are conflicting in nature. This paper presents an application of fuzzy goal programming to the linear multiobjective transportation problem. In this paper, we use a special type of nonlinear (hyperbolic and exponential) membership functions to solve multiobjective transportation problem. It gives an optimal compromise solution. The obtained result has been compared with the solution obtained by using a linear membership function. To illustrate the methodology some numerical examples are presented.

1. Introduction

The transportation problem is one of the earliest applications of linear programming problems. The basic transportation problem was originally developed by Hitchcock [17]. Subsequently, several kinds of transportation problems have appeared in the literature [4, 19, 22, 27]. Efficient methods of solution derived from the simplex algorithm were developed in 1947, primarily by Dantzig [9] and then by Charnes et al. [7]. The transportation problem can be modeled as a standard linear programming problem, which can then be solved by the simplex method. Lee et al. [22] have studied the optimization of transportation problems with multiple objectives. Diaz [12] and Isermann [18] have developed algorithms for identifying all of the nondominated solutions for a linear multiobjective transportation problem. Current et al. [8] have done a review of multiobjective design of transportation networks. Diaz [11] presents an alternative procedure to generate all nondominated solutions to the multiobjective transportation problem. This approach depends upon specifying a priori measure of the closeness of any compromise solution to the ideal solution. Edwards [13] advocates the use of an additive multiattribute utility function for the linear multiobjective problem. An additive linear function may provide a good initial solution for the linear multiobjective problem. Ringuest et al. [27] have developed two interactive algorithms for the linear multiobjective transportation problem. Bit et al. [3] applied the fuzzy programming technique with linear membership function to solve multiobjective transportation problem

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(MOTP), and obtained efficient solutions for MOTP as well as an optimal compromise solution. Li et al. [23] presented a fuzzy approach to the MOTP. Verma et al. [30] applied a fuzzy programming technique to solve MOTP with some nonlinear membership functions. Other research works in this realm presented in [20, 26].

In 1961, goal programming introduced by Charnes and Cooper [6]. Aenaida and Kwak [2] applied goal programming to find a solution for MOTP. Recently, the authors used the fuzzy goal programming approach to solve MOTP [32]. Other authors used fuzzy goal programming technique to solve different types of multiobjective linear programming problems [1, 14, 15, 16, 24, 25, 29, 31].

Leberling [21] used hyperbolic membership function for the multiobjective linear programming problem. He showed that solutions obtained by fuzzy linear programming with this type of nonlinear membership function are always efficient. Dhingra and Moskowitz [10] defined other types of the nonlinear (exponential, quadratic and logarithmic) membership functions and applied them to an optimal design problem. Peidro and Vasant [26] adopted a nonlinear membership function known as the modified S-curve. They used modified S-curve membership function to apply an interactive fuzzy approach to solve the multi-objective transportation planning decision problem for the purpose of finding a preferred compromise solution. We apply the fuzzy goal programming technique with some nonlinear (hyperbolic and exponential) membership functions to solve multiobjective transportation problems. The paper has the following structure. Section 2 reviews the problem formulation. In section 3, we study some nonlinear (hyperbolic and exponential) membership functions. Section 4 uses fuzzy goal programming approach with some nonlinear (hyperbolic and exponential) membership functions to solve MOTP. In section 5, some examples are presented.

2. Problem Formulation

In the real-world situations, the transportation problem usually involves multiple, incommensurable and conflicting objective functions. This kind of problem is called multiobjective transportation problem. Similar to a typical transportation problem in a MOTP a product is to be transported from m sources to n destinations and their capacities are $a_1, a_2, ..., a_m$ and $b_1, b_2, ..., b_n$, respectively. In addition, there is a penalty c_{ij} associated with transporting a unit of product from source i to destination j. The penalty may be cost or delivery time or safety of delivery or etc. A variable x_{ij} represents the unknown quantity to be shipped from source i to destination j. A mathematical model of MOTP can be writhen as follows:

$$\min\ Z_r = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^r x_{ij}, \quad r = 1, 2, \cdots, k,$$

s.t.

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i, \quad i = 1, 2, \cdots, m, \\ \sum_{i=1}^{m} x_{ij} &= b_j, \quad j = 1, 2, \cdots, n, \\ x_{ij} &\geq 0, \quad \text{for all } i, j. \end{aligned} \tag{1}$$

The subscript on Z_r and superscript on c_{ij}^r are related to the rth penalty criterion. Without loss of generality, it will be assumed that $a_i \geq 0$ for all i, $b_j \geq 0$ for all j and the equilibrium condition $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ is satisfied.

We denote by **S** the set of all feasible solutions of the MOTP, i.e.,

$$\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^{m \times n} \mid \sum_{j=1}^{n} x_{ij} = a_i,\ \sum_{i=1}^{m} x_{ij} = b_j,\ x_{ij} \geq 0,$$
$$i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n\}.$$

We review the concept of optimality for MOTP as usual manner.

Definition 2.1. A feasible solution $\mathbf{x}^* = \{x_{ij}{}^*\} \in \mathbf{S}$ is an efficient (nondominated) solution for MOTP if and only if there does not exist another $\mathbf{x} = \{x_{ij}\} \in \mathbf{S}$ such that $Z_r(\mathbf{x}) \leq Z_r(\mathbf{x}^*),\ r = 1, 2, ..., k$, and $Z_l(\mathbf{x}) \neq Z_l(\mathbf{x}^*)$, for some $l, 1 \leq l \leq k$.

Definition 2.2. A feasible solution $\mathbf{x}^* = \{x_{ij}{}^*\} \in \mathbf{S}$ is a weak efficient solution for MOTP if and only if there does not exist another $\mathbf{x} = \{x_{ij}\} \in \mathbf{S}$ such that $Z_r(\mathbf{x}) < Z_r(\mathbf{x}^*),\ r = 1, 2, ..., k$.

Definition 2.3. [3] A feasible solution $\mathbf{x}^* = \{x_{ij}{}^*\} \in \mathbf{S}$ is an optimal compromise solution for MOTP if it is preferred by DM to all other feasible solutions, taking into consideration all criteria contained in the multiobjective functions.

Let E and E^w denote the set of all efficient solutions and all weak efficient solutions for MOTP, respectively, then $E \subseteq E^w$. Note that an optimal compromise solution of MOTP has to be a weak efficient solution of MOTP, according to the definition of weak efficient solutions.

3. Membership Functions

One of the major assumptions in solving fuzzy mathematical programming problems in the literature involves the use of linear membership functions for all fuzzy sets involved in a decision making process. A linear approximation is most commonly used because of its simplicity and is defined by fixing two points, the upper and lower levels of acceptability. If fuzzy set theory is to be considered a purely formal theory, such an assumption is acceptable, even though some kind of formal justification of this assumption would be desirable. If, however, fuzzy set theory is used to model real decision making processes, and an assertion is made that the resulting models are true models of reality, then some kind of empirical justification for this assumption is necessary. In view of this, several other (nonlinear) shapes for membership functions, such as concave or convex shaped membership functions are analyzed to determine their impact on the overall design process.

Let L_r and U_r be the aspired level of achievement and the highest acceptable level of achievement for the r-th objective function, respectively. In the following three subsections we study different membership functions.

3.1. **Linear Membership Function.** A linear membership function can be defined as follows.

$$\mu_r(Z_r(\mathbf{x})) = \begin{cases} 1 & \text{if } Z_r \leq L_r, \\ 1 - \frac{Z_r - L_r}{U_r - L_r} & \text{if } L_r < Z_r < U_r, \\ 0 & \text{if } Z_r \geq U_r. \end{cases} \quad (2)$$

3.2. **Exponential Membership Function.** An exponential membership function is defined by

$$\mu_r^E(Z_r(\mathbf{x})) = \begin{cases} 1 & \text{if } Z_r \leq L_r, \\ \frac{e^{-s\psi_r(x)} - e^{-s}}{1 - e^{-s}} & \text{if } L_r \leq Z_r \leq U_r, \\ 0 & \text{if } Z_r \geq U_r, \end{cases} \quad (3)$$

where $\psi_r(x) = (Z_r - L_r)/(U_r - L_r),\ r = 1, 2, \ldots, k$ and s is a non-zero parameter prescribed by the decision maker. Figure 1 depicts a possible shape of $\mu_r^E(Z_r(\mathbf{x}))$ with respect to $Z_r(\mathbf{x})$ [10].

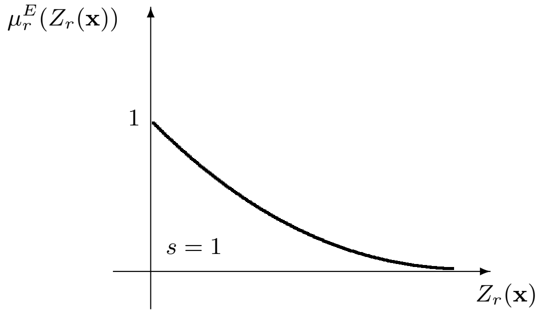

FIGURE 1. Exponential Membership Function

3.3. **Hyperbolic Membership Function.** The hyperbolic function [28] is convex over a part of the objective function values and is concave over the remaining part. The rationale for such a shape has been discussed in [5] for utility functions and in our problem context is as follows: When the decision maker is worse off with respect to a goal, the decision maker tends to have a higher marginal rate of satisfaction with respect to that goal. A convex shape captures that behavior in the membership function. On the other hand, when one is better off with respect to a goal, one tends to have a smaller marginal rate of satisfaction. Such behavior is modeled using the concave portion of the membership function. The complete function is as follows:

$$\mu_r^H(Z_r(\mathbf{x})) = \begin{cases} 1 & \text{if } Z_r \leq L_r, \\ \frac{1}{2}\tanh(\frac{U_r + L_r}{2} - Z_r(\mathbf{x})\}\alpha_r) + \frac{1}{2} & \text{if } L_r \leq Z_r \leq U_r, \\ 0 & \text{if } Z_r \geq U_r, \end{cases} \quad (4)$$

where $\alpha_r = \frac{6}{U_r - L_r}$.

This membership function has the following formal properties [33]:

(1) $\mu_r^H(Z_r(\mathbf{x}))$ is strictly monotonously decreasing function with respect to $Z_r(\mathbf{x})$;

(2) $\mu_r^H(Z_r(\mathbf{x})) = \frac{1}{2} \;\Leftrightarrow\; Z_r(\mathbf{x}) = \frac{1}{2}(U_r + L_r)$;

(3) $\mu_r^H(Z_r(\mathbf{x}))$ is strictly convex for $Z_r(\mathbf{x}) \geq \frac{1}{2}(U_r+L_r)$ and strictly concave for $Z_r(\mathbf{x}) \leq \frac{1}{2}(U_r + L_r)$;

(4) $\mu_r^H(Z_r(\mathbf{x}))$ satisfies $0 < \mu_r^H(Z_r(\mathbf{x})) < 1$ for $L_r < Z_r(\mathbf{x})) < U_r$ and approaches asymptotically $\mu_r^H(Z_r(\mathbf{x})) = 0$ and $\mu_r^H(Z_r(\mathbf{x})) = 1$ as $Z_r(\mathbf{x}) \to \infty$ and $-\infty$, respectively.

Figure 2 depicts a possible shape of $\mu_r^H(Z_r(\mathbf{x}))$ with respect to $Z_r(\mathbf{x})$ [28].

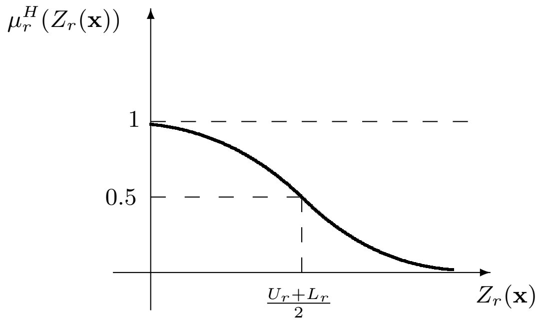

Figure 2. Hyperbolic Membership Function

4. **Fuzzy Goal Programming Approach for Solving MOTP**

Mohamed in [24] has used linear membership functions, where he introduced fuzzy goal programming approach for solving multiobjective linear programming problem. In [32], mohamed's approach was adopted to present a fuzzy goal programming approach for solving multiobjective transportation problems.

Let L_r and U_r be the aspired level of achievement and the highest acceptable level of achievement for the r-th objective function, respectively.

To solve MOTP problem based on the fuzzy goal programming technique [32], one can use the following steps:

Step 1: Solve the multiobjective transportation problem as a single objective transportation problem, taking each time only one objective as objective function and ignoring all others.

Step 2: Compute the value of each objective function at each solution derived in Step 1.

Step 3: From Step 2, find for each objective the best (L_r) and the worst (U_r) values corresponding to the set of solutions. Recall that L_r and U_r are the aspired level of achievement and the highest acceptable level of achievement for the r-th objective function, respectively.

Step 4: Define a membership functions μ_r (linear μ_r^L, hyperbolic μ_r^H or exponential μ_r^E) for the rth objective function.

If we use the linear membership function as defined in (2) then an equivalent linear model for the model (1) can be formulated as:

$$
\begin{aligned}
&\min : \phi, \\
&\text{s.t.} \\
&\frac{U_r - Z_r}{U_r - L_r} + d_r^- - d_r^+ = 1, \\
&\phi \geq d_r^-, \quad r = 1, 2, \cdots, k, \\
&d_r^+ d_r^- = 0, \\
&\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \cdots, m, \\
&\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \cdots, n, \\
&d_r^+, \; d_r^- \geq 0, \\
&\phi \leq 1, \; \phi \geq 0, \\
&x_{ij} \geq 0, \quad \text{for all } i, j,
\end{aligned}
$$

where the equilibrium condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is satisfied.

If we use the exponential membership function as defined in (3), then an equivalent nonlinear model for the model (1) can be formulated as:

$$
\begin{aligned}
&\min : \phi, \\
&\text{s.t.} \\
&\frac{e^{-s\psi_r(x)} - e^{-s}}{1 - e^{-s}} + d_r^- - d_r^+ = 1, \\
&\phi \geq d_r^-, \quad r = 1, 2, \cdots, k, \\
&d_r^+ d_r^- = 0, \\
&\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \cdots, m, \\
&\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \cdots, n, \\
&d_r^+, \; d_r^- \geq 0, \\
&\phi \leq 1, \; \phi \geq 0, \\
&x_{ij} \geq 0, \quad \text{for all } i, j,
\end{aligned}
$$

where the equilibrium condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is satisfied.

If we use the hyperbolic membership function as defined in (4) then an equivalent nonlinear model for the model (1) can be formulated as:

$$\begin{aligned}
&\min : \phi,\\
&\text{s.t.}\\
&\frac{1}{2}+\frac{1}{2}\frac{e^{\{\frac{(U_r+L_r)}{2}-Z_r\}\alpha_r}-e^{-\{\frac{(U_r+L_r)}{2}-Z_r\}\alpha_r}}{e^{\{\frac{(U_r+L_r)}{2}-Z_r\}\alpha_r}+e^{-\{\frac{(U_r+L_r)}{2}-Z_r\}\alpha_r}}+d_r^- - d_r^+ = 1,\\
&\phi \geq d_r^-, \quad r=1,2,\cdots,k,\\
&d_r^+ d_r^- = 0,\\
&\sum_{j=1}^{n} x_{ij} = a_i, \quad i=1,2,\cdots,m,\\
&\sum_{i=1}^{m} x_{ij} = b_j, \quad j=1,2,\cdots,n,\\
&d_r^+,\ d_r^- \geq 0,\\
&\phi \leq 1,\ \phi \geq 0,\\
&x_{ij} \geq 0, \quad \text{for all } i,j,
\end{aligned}$$

where the equilibrium condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is satisfied.

Step 5: Solve the equivalent crisp model obtained in Step 4.
The solution obtained in Step 5 will be the optimal compromise solution of MOTP model [30].

5. **Application Examples**

Example 5.1. To illustrate the efficiency of the proposed method, we consider the following numerical example presented by Verma et al. [30]:

$$\begin{aligned}
\min\ Z_1 =& 16x_{11}+19x_{12}+12x_{13}+22x_{21}+13x_{22}+\\
& 19x_{23}+14x_{31}+28x_{32}+8x_{33},\\
\min\ Z_2 =& Z_2 = 9x_{11}+14x_{12}+12x_{13}+16x_{21}+10x_{22}+\\
& 14x_{23}+8x_{31}+20x_{32}+6x_{33}
\end{aligned}$$

$$\begin{aligned}
\text{s.t.}\quad & \\
& x_{11}+x_{12}+x_{13}=14,\\
& x_{21}+x_{22}+x_{23}=16,\\
& x_{31}+x_{32}+x_{33}=12,\\
& x_{11}+x_{21}+x_{31}=10,\\
& x_{12}+x_{22}+x_{32}=15,\\
& x_{13}+x_{23}+x_{33}=17,\\
& x_{ij}\geq 0, \quad i=1,2,3, \quad j=1,2,3.
\end{aligned}$$

In the following the proposed steps of the previous section are presented.

Step 1: The solution of each single objective transportation problem is:

$$X^1 = (x_{11}^1 = 9, x_{12}^1 = 0, x_{13}^1 = 5, x_{21}^1 = 1, x_{22}^1 = 5, x_{23}^1 = 0, x_{31}^1 = 0, x_{32}^1 = 0, x_{33}^1 = 12)',$$
$$X^2 = (x_{11}^2 = 10, x_{12}^2 = 0, x_{13}^2 = 4, x_{21}^2 = 0, x_{22}^2 = 15, x_{23}^2 = 1, x_{31}^2 = 0, x_{32}^2 = 0, x_{33}^2 = 12)'.$$

Step 2: The objective function values are:

$$Z_1(X^1) = 517,\ Z_1(X^2) = 518,\ Z_2(X^1) = 379, Z_2(X^2) = 374.$$

Step 3: The upper and lower bounds of each objective function can be written as follows:

$$517 \le Z_1 \le 518,\ 374 \le Z_2 \le 379,$$
$$L_1 = 517,\ U_1 = 518,\ L_2 = 374,\ U_2 = 379.$$

Step 4: If we use the linear membership function as defined in (2), an equivalent crisp model can be formulated as:

$$\begin{aligned}
&\min : \phi \\
&\text{s.t.} \\
&\quad 518 - Z_1 + d_1^- - d_1^+ = 1, \\
&\quad \frac{379 - Z_2}{5} + d_2^- - d_2^+ = 1, \\
&\quad \phi \ge d_r^-, \quad r = 1, 2, \\
&\quad d_r^+ d_r^- = 0, \\
&\quad x_{11} + x_{12} + x_{13} = 14, \\
&\quad x_{21} + x_{22} + x_{23} = 16, \\
&\quad x_{31} + x_{32} + x_{33} = 12, \\
&\quad x_{11} + x_{21} + x_{31} = 10, \\
&\quad x_{12} + x_{22} + x_{32} = 15, \\
&\quad x_{13} + x_{23} + x_{33} = 17, \\
&\quad d_r^+,\ d_r^- \ge 0, \\
&\quad \phi \le 1,\ \phi \ge 0, \\
&\quad x_{ij} \ge 0, \quad \text{for all } i, j,
\end{aligned}$$

where $Z_1 = 16x_{11} + 19x_{12} + 12x_{13} + 22x_{21} + 13x_{22} + 19x_{23} + 14x_{31} + 28x_{32} + 8x_{33}$ and $Z_2 = 9x_{11} + 14x_{12} + 12x_{13} + 16x_{21} + 10x_{22} + 14x_{23} + 8x_{31} + 20x_{32} + 6x_{33}$.

The problem is solved and the results are:

$$x_{11}^* = 9.5,\ x_{13}^* = 4.5,\ x_{21}^* = 0.5, x_{22}^* = 15,\ x_{23}^* = 0.5,\ x_{33}^* = 12,$$
$$d_1^- = 0.5,\ d_1^+ = 0,\ d_2^- = 0.5,\ d_2^+ = 0,\ \phi^* = 0.5,$$
$$Z_1^* = 517.5,\ Z_2^* = 376.5.$$

The other variables that are not in the above have a zero value.

If we use the exponential membership function as defined in (3) with the parameter $s = 1$, an equivalent crisp model can be formulated as:

$$\begin{aligned}
&\min : \phi \\
&\text{s.t.} \\
&\quad \frac{e^{-(Z_1-517)} - e^{-1}}{1-e^{-1}} + d_1^- - d_1^+ = 1, \\
&\quad \frac{e^{-(Z_2-374)/5} - e^{-1}}{1-e^{-1}} + d_2^- - d_2^+ = 1, \\
&\quad \phi \geq d_r^-, \quad r = 1, 2, \\
&\quad d_r^+ d_r^- = 0, \\
&\quad x_{11} + x_{12} + x_{13} = 14, \\
&\quad x_{21} + x_{22} + x_{23} = 16, \\
&\quad x_{31} + x_{32} + x_{33} = 12, \\
&\quad x_{11} + x_{21} + x_{31} = 10, \\
&\quad x_{12} + x_{22} + x_{32} = 15, \\
&\quad x_{13} + x_{23} + x_{33} = 17, \\
&\quad d_r^+, \ d_r^- \geq 0, \\
&\quad \phi \leq 1, \ \phi \geq 0, \\
&\quad x_{ij} \geq 0, \quad \text{for all } i, j,
\end{aligned}$$

where $Z_1 = 16x_{11} + 19x_{12} + 12x_{13} + 22x_{21} + 13x_{22} + 19x_{23} + 14x_{31} + 28x_{32} + 8x_{33}$ and $Z_2 = 9x_{11} + 14x_{12} + 12x_{13} + 16x_{21} + 10x_{22} + 14x_{23} + 8x_{31} + 20x_{32} + 6x_{33}$.
The problem is solved and the results are:

$$\begin{aligned}
&x_{11}^* = 9.5, \ x_{13}^* = 4.5, \ x_{21}^* = 0.5, x_{22}^* = 15, \ x_{23}^* = 0.5, \ x_{33}^* = 12, \\
&d_1^- = 0.62, \ d_1^+ = 0, \ d_2^- = 0.62, \ d_2^+ = 0, \ \phi^* = 0.62, \\
&Z_1^* = 517.5, \ Z_2^* = 376.5,
\end{aligned}$$

and the other variables that are not in the above have a zero value.

If we use the hyperbolic membership function as defined in (4) then an equivalent crisp model can be formulated as:

$$\begin{aligned}
&\min : \phi \\
&\text{s.t.} \\
&\quad \frac{1}{2} + \frac{1}{2}\frac{e^{6(517.5-Z_1)} - e^{-6(517.5-Z_1)}}{e^{6((517.5-Z_1)} + e^{-6(517.5-Z_1)}} + d_1^- - d_1^+ = 1, \\
&\quad \frac{1}{2} + \frac{1}{2}\frac{e^{\frac{6}{5}(376.5-Z_2)} - e^{-\frac{6}{5}(376.5-Z_2)}}{e^{\frac{6}{5}(376.5-Z_2)} + e^{-\frac{6}{5}(376.5-Z_2)}} + d_2^- - d_2^+ = 1, \\
&\quad \phi \geq d_r^-, \quad r = 1, 2, \\
&\quad d_r^+ d_r^- = 0, \\
&\quad x_{11} + x_{12} + x_{13} = 14, \\
&\quad x_{21} + x_{22} + x_{23} = 16, \\
&\quad x_{31} + x_{32} + x_{33} = 12, \\
&\quad x_{11} + x_{21} + x_{31} = 10, \\
&\quad x_{12} + x_{22} + x_{32} = 15, \\
&\quad x_{13} + x_{23} + x_{33} = 17, \\
&\quad d_r^+, \ d_r^- \geq 0, \\
&\quad \phi \leq 1, \ \phi \geq 0, \\
&\quad x_{ij} \geq 0, \quad \text{for all } i, j,
\end{aligned}$$

where $Z_1 = 16x_{11} + 19x_{12} + 12x_{13} + 22x_{21} + 13x_{22} + 19x_{23} + 14x_{31} + 28x_{32} + 8x_{33}$ and $Z_2 = 9x_{11} + 14x_{12} + 12x_{13} + 16x_{21} + 10x_{22} + 14x_{23} + 8x_{31} + 20x_{32} + 6x_{33}$.

The problem is solved and the results are:

$$x^*_{11} = 9.5,\ x^*_{13} = 4.5,\ x^*_{21} = 0.5, x^*_{22} = 15,\ x^*_{23} = 0.5,\ x^*_{33} = 12,$$
$$d^-_1 = 0.5,\ d^+_1 = 0,\ d^-_2 = 0.5,\ d^+_2 = 0,\ \phi^* = 0.5,$$
$$Z^*_1 = 517.5,\ Z^*_2 = 376.5.$$

The other variables that are not in the above have a zero value.

Example 5.2. This example is adopted from Diaz [11].

$$\begin{aligned}
\min\ Z_1 =&9x_{11} + 12x_{12} + 9x_{13} + 6x_{14} + 9x_{15} + 7x_{21} + 3x_{22} + 7x_{23} + 7x_{24} + 5x_{25}\\
&+ 6x_{31} + 5x_{32} + 9x_{33} + 11x_{34} + 3x_{35} + 6x_{41} + 8x_{42} + 11x_{43} + 2x_{44} + 2x_{45},\\
\min\ Z_2 =&2x_{11} + 9x_{l2} + 8x_{13} + x_{l4} + 4x_{15} + x_{21} + 9x_{22} + 9x_{23} + 5x_{24} + 2x_{25},\\
&+ 8x_{31} + x_{32} + 8x_{33} + 4x_{34} + 5x_{35} + 2x_{41} + 8x_{42} + 6x_{43} + 9x_{44} + 8x_{45},\\
\min\ Z_3 =&2x_{11} + 4x_{12} + 6x_{13} + 3x_{14} + 6x_{15} + 4x_{21} + 8x_{22} + 4x_{23} + 9x_{24} + 2x_{25}\\
&+ 5x_{31} + 3x_{32} + 5x_{33} + 3x_{34} + 6x_{35} + 6x_{41} + 9x_{42} + 6x_{43} + 3x_{44} + x_{45},
\end{aligned}$$

s.t.

$$\begin{aligned}
&x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 5,\\
&x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 4,\\
&x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 2,\\
&x_{41} + x_{42} + x_{43} + x_{44} + x_{45} = 9,\\
&x_{11} + x_{21} + x_{31} + x_{41} = 4,\\
&x_{12} + x_{22} + x_{32} + x_{42} = 4,\\
&x_{13} + x_{23} + x_{33} + x_{43} = 6,\\
&x_{14} + x_{24} + x_{34} + x_{44} = 2,\\
&x_{15} + x_{25} + x_{35} + x_{45} = 4,\\
&x_{ij} \geq 0, \quad i = 1, 2, 3,4, \quad j = 1, 2, 3, 4, 5.
\end{aligned}$$

In the following the proposed steps of the previous section are presented.

Step 1: The solution of each single objective transportation problem is:

$X^1 = (x^1_{13} = 5, x^1_{22} = 3, x^1_{23} = 1, x^1_{31} = 1, x^1_{32} = 1, x^1_{41} = 3, x^1_{44} = 2, x^1_{45} = 4)'$

$X^2 = (x^2_{11} = 3, x^2_{14} = 2, x^2_{25} = 4, x^2_{32} = 2, x^2_{41} = 1, x^2_{42} = 2, x^2_{43} = 6)'$

$X^3 = (x^3_{11} = 3, x^3_{12} = 2, x^3_{21} = 1, x^3_{23} = 3, x^3_{32} = 2, x^3_{43} = 3, x^3_{44} = 2, x^3_{45} = 4)'$.

Step 2: The objective function values are:

$Z_1(X^1) = 102,\ Z_1(X^2) = 157,\ Z_1(X^3) = 134,\ Z_2(X^1) = 141,\ Z_2(X^2) = 72,\ Z_2(X^3) = 116,\ Z_3(X^1) = 94,\ Z_3(X^2) = 86,\ Z_3(X^3) = 64.$

Step 3: The upper and lower bounds of each objective function can be written as follows:

$$102 \leq Z_1 \leq 157,\ 72 \leq Z_2 \leq 141, \text{ and } 64 \leq Z_3 \leq 94,$$

hence $L_1 = 102,\ U_1 = 157,\ L_2 = 72,\ U_2 = 141,\ L_3 = 64$, and $U_3 = 94$.

Step 4: If we use the linear membership function as defined in (2), an equivalent crisp model can be formulated as:

$$
\begin{aligned}
&\min : \phi \\
&\text{s.t.} \\
&\quad \frac{157 - Z_1}{55} + d_1^- - d_1^+ = 1, \\
&\quad \frac{141 - Z_2}{69} + d_2^- - d_2^+ = 1, \\
&\quad \frac{94 - Z_3}{30} + d_3^- - d_3^+ = 1, \\
&\quad \phi \geq d_r^-, \quad r = 1, 2, 3, \\
&\quad d_r^+ d_r^- = 0, \\
&\quad x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 5, \\
&\quad x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 4, \\
&\quad x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 2, \\
&x_{41} + x_{42} + x_{43} + x_{44} + x_{45} = 9, \\
&x_{11} + x_{21} + x_{31} + x_{41} = 4, \\
&x_{12} + x_{22} + x_{32} + x_{42} = 4, \\
&x_{13} + x_{23} + x_{33} + x_{43} = 6, \\
&x_{14} + x_{24} + x_{34} + x_{44} = 2, \\
&x_{15} + x_{25} + x_{35} + x_{45} = 4, \\
&d_r^+, \ d_r^- \geq 0, \\
&\phi \leq 1, \ \phi \geq 0, \\
&x_{ij} \geq 0, \quad \text{for all } i, j,
\end{aligned}
$$

where $Z_1 = 9x_{11} + 12x_{12} + 9x_{13} + 6x_{14} + 9x_{15} + 7x_{21} + 3x_{22} + 7x_{23} + 7x_{24} + 5x_{25} + 6x_{31} + 5x_{32} + 9x_{33} + 11x_{34} + 3x_{35} + 6x_{41} + 8x_{42} + 11x_{43} + 2x_{44} + 2x_{45}$, $Z_2 = 2x_{11} + 9x_{l2} + 8x_{13} + x_{l4} + 4x_{15} + x_{21} + 9x_{22} + 9x_{23} + 5x_{24} + 2x_{25}, +8x_{31} + x_{32} + 8x_{33} + 4x_{34} + 5x_{35} + 2x_{41} + 8x_{42} + 6x_{43} + 9x_{44} + 8x_{45}$, and $Z_3 = 2x_{11} + 4x_{12} + 6x_{13} + 3x_{14} + 6x_{15} + 4x_{21} + 8x_{22} + 4x_{23} + 9x_{24} + 2x_{25} + 5x_{31} + 3x_{32} + 5x_{33} + 3x_{34} + 6x_{35} + 6x_{41} + 9x_{42} + 6x_{43} + 3x_{44} + x_{45}$.

The problem is solved and the results are:

$$
\begin{aligned}
&x_{11}^* = 2.737554, \ x_{13}^* = 0.2624456, \ x_{14} = 2.000000, \ x_{22}^* = 2.000000, \ x_{23}^* = 1.842114, \\
&x_{25}^* = 0.1578863, \ x_{32}^* = 2.000000, \ x_{41}^* = 1.262446, \ x_{43}^* = 3.895441, \ x_{45}^* = 3.842114, \\
&d_1^- = 0.4507814, , \ d_1^+ = 0, \ d_2^- = 0.4507814, \ d_2^+ = 0, \ d_3^- = 0.4507814, \ d_3^+ = 0, \ \phi^* = 0.4507814, \\
&Z_1^* = 126.7930, \ Z_2^* = 103.1039, \ Z_3^* = 77.52344,
\end{aligned}
$$

and the other variables that are not in the above have a zero value.

If we use the exponential membership function as defined in (3) with the parameter $s = 1$, an equivalent crisp model can be formulated as:

$$
\begin{aligned}
&\min : \phi \\
&\text{s.t.} \\
&\quad \frac{e^{-(Z_1-102)/55} - e^{-1}}{1 - e^{-1}} + d_1^- - d_1^+ = 1, \\
&\quad \frac{e^{-(Z_2-72)/69} - e^{-1}}{1 - e^{-1}} + d_2^- - d_2^+ = 1, \\
&\quad \frac{e^{-(Z_3-64)/30} - e^{-1}}{1 - e^{-1}} + d_3^- - d_3^+ = 1, \\
&\quad \phi \geq d_r^-, \quad r = 1, 2, 3, \\
&\quad d_r^+ d_r^- = 0, \\
&\quad x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 5, \\
&\quad x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 4, \\
&\quad x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 2, \\
&\quad x_{41} + x_{42} + x_{43} + x_{44} + x_{45} = 9, \\
&\quad x_{11} + x_{21} + x_{31} + x_{41} = 4,
\end{aligned}
$$

$$
\begin{aligned}
& x_{12}+x_{22}+x_{32}+x_{42}=4, \\
& x_{13}+x_{23}+x_{33}+x_{43}=6, \\
& x_{14}+x_{24}+x_{34}+x_{44}=2, \\
& x_{15}+x_{25}+x_{35}+x_{45}=4, \\
& d_r^+,\ d_r^- \geq 0, \\
& \phi \leq 1,\ \phi \geq 0, \\
& x_{ij} \geq 0, \quad \text{for all } i, j.
\end{aligned}
$$

The problem is solved and the results are:

$x_{11}^* = 2.737554,\ x_{13}^* = 0.2624456,\ x_{14} = 2.000000,\ x_{22}^* = 2.000000,\ x_{23}^* = 1.842114,$
$x_{25}^* = 0.1578863,\ x_{32}^* = 2.000000,\ x_{41}^* = 1.262446,\ x_{43}^* = 3.895441,\ x_{45}^* = 3.842114,$
$d_1^- = 0.5740517,\ d_1^+ = 0,\ d_2^- = 0.5740517,\ d_2^+ = 0,\ d_3^- = 0.5740517,\ d_3^+ = 0,\ \phi^* = 0.5740517,$
$Z_1^* = 126.7930,\ Z_2^* = 103.1039,\ Z_3^* = 77.52344.$

The other variables that are not in the above have a zero value.

If we use the hyperbolic membership function as defined in (4) then an equivalent crisp model can be formulated as:

$$
\begin{aligned}
& \min : \phi \\
& \text{s.t.} \\
& \frac{1}{2}+\frac{1}{2}\frac{e^{\frac{6}{55}(129.5-Z_2)}-e^{-\frac{6}{55}(129.5-Z_2)}}{e^{\frac{6}{55}(129.5-Z_2)}+e^{-\frac{6}{55}(129.5-Z_2)}}+d_1^- - d_1^+ = 1, \\
& \frac{1}{2}+\frac{1}{2}\frac{e^{\frac{6}{69}(106.5-Z_2)}-e^{-\frac{6}{69}(106.5-Z_2)}}{e^{\frac{6}{69}(106.5-Z_2)}+e^{-\frac{6}{69}(106.5-Z_2)}}+d_2^- - d_2^+ = 1, \\
& \frac{1}{2}+\frac{1}{2}\frac{e^{\frac{6}{30}(79-Z_3)}-e^{-\frac{6}{30}(79-Z_3)}}{e^{\frac{6}{30}(79-Z_3)}+e^{-\frac{6}{30}(79-Z_3)}}+d_2^- - d_2^+ = 1, \\
& \phi \geq d_r^-, \quad r=1,2,3, \\
& d_r^+ d_r^- = 0, \\
& x_{11}+x_{12}+x_{13}+x_{14}+x_{15}=5, \\
& x_{21}+x_{22}+x_{23}+x_{24}+x_{25}=4, \\
& x_{31}+x_{32}+x_{33}+x_{34}+x_{35}=2, \\
& x_{41}+x_{42}+x_{43}+x_{44}+x_{45}=9, \\
& x_{11}+x_{21}+x_{31}+x_{41}=4, \\
& x_{12}+x_{22}+x_{32}+x_{42}=4, \\
& x_{13}+x_{23}+x_{33}+x_{43}=6, \\
& x_{14}+x_{24}+x_{34}+x_{44}=2, \\
& x_{15}+x_{25}+x_{35}+x_{45}=4, \\
& d_r^+,\ d_r^- \geq 0, \\
& \phi \leq 1,\ \phi \geq 0, \\
& x_{ij} \geq 0, \quad \text{for all } i, j.
\end{aligned}
$$

The problem is solved and the results are:

$x_{11}^* = 2.737554,\ x_{13}^* = 0.2624456,\ x_{14} = 2.000000,\ x_{22}^* = 2.000000,\ x_{23}^* = 1.842114,$
$x_{25}^* = 0.1578863,\ x_{32}^* = 2.000000,\ x_{41}^* = 1.262446,\ x_{43}^* = 3.895441,\ x_{45}^* = 3.842114,$
$d_1^- = 0.3564918,\ d_1^+ = 0,\ d_2^- = 0.3564918,\ d_2^+ = 0,\ d_3^- = 0.3564918,\ d_3^+ = 0,\ \phi^* = 0.3564918,$
$Z_1^* = 126.7930,\ Z_2^* = 103.1039,\ Z_3^* = 77.52344,$

and the other variables that are not in the above have a zero value.

6. Conclusion

In this paper, three special types of membership functions have been used to solve the multi-objective transportation problem. The optimal compromise solution does not change if we compare with the solution obtained by the linear membership function. However, if we use the exponential membership function, with different values of s (parameter) then the optimal compromise solution does not change significantly, if we compare with the solution obtained by the linear membership function. Further, we conclude that for a multi-objective probabilistic transportation problem if the demand parameters are gamma random variables, then the deterministic problem becomes non-linear. To solve this type of problem, these nonlinear membership functions can be used. Apart from the transportation problems for the multiobjective nonlinear programming problems, nonlinear membership functions are useful.

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MINIMIZATION OF DETERMINISTIC FINITE AUTOMATA WITH VAGUE (FINAL) STATES AND INTUITIONISTIC FUZZY (FINAL) STATES

A. CHOUBEY AND K. M. RAVI

Abstract. In this paper, relations among the membership values of generalized fuzzy languages such as intuitionistic fuzzy language, interval-valued fuzzy language and vague language are studied. It will aid in studying the properties of one language when the properties of another are known.

Further, existence of a minimized finite automaton with vague (final) states for any vague regular language recognized by a finite automaton with vague (final) states is shown in this paper. Finally, an efficient algorithm is given for minimizing the finite automaton with vague (final) states. Similarly, it can be shown for intuitionistic fuzzy regular language. These may contribute to a better understanding of the role of finite automaton with vague (final) states or the finite automaton with intuitionistic fuzzy (final) states while studying lexical analysis, decision making etc.

1. Introduction

Fuzziness reduces the gap between formal language and natural language in terms of precision, leading to describe fuzzy language. Fuzzy language and fuzzy grammars were formerly defined by Lee and Zadeh [12]. A fuzzy language $\tilde{L}$ in the set of finite alphabet Σ, is a class of strings $w \in \Sigma^*$ along with a grade of membership function $f_{\tilde{L}}(w)$. This membership function assigns to each string a grade of membership value in [0, 1]. Fuzzy language is further generalized as intuitionistic fuzzy language (IFL) [17], interval-valued fuzzy language (IVFL) [18] and vague language (VL) [6] using the notion of intuitionistic fuzzy sets [1], [2], interval-valued fuzzy sets [8] and vague sets [7] respectively. Our motive is to study the membership values of these languages in a generalized set up. Here, we have shown that there is a relation between the membership values of the strings in IFL, IVFL and VL respectively.

One of the most significant branches of the algebraic theory of languages and automata is Myhill-Nerode's theory [9], where recognizability of regular languages by finite automata is studied through right invariant equivalence classes. Also, it is a powerful tool for minimizing the number of redundant states in a finite automaton. Myhill-Nerode's theorem has been extended to fuzzy regular language and also an

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algorithm is given for minimizing the deterministic finite automaton with fuzzy (final) states in [13], [15]. Since finite automata constitutes a mathematical model on computation, fuzzy finite automata may be considered as an extended model which includes notions like vagueness and imprecision frequently encountered in the study of fuzzy language. The models of general fuzzy automata and general fuzzy recognizers are given in [10] and [11] respectively.

In this paper, we have shown that for any vague regular language and for any intuitionistic fuzzy regular language recognizable respectively by vague (final) states automaton and intuitionistic fuzzy (final) states automaton, there exists a minimal vague (final) states automaton and a minimal intuitionistic fuzzy (final) states automaton. These automata are unique up to an isomorphism. Furthermore, an efficient algorithm is given for minimizing vague (final) states automaton and intuitionistic fuzzy (final) states automaton. These may help out to a better understanding of the role of vague (final) states automaton or intuitionistic fuzzy (final) states automaton while studying lexical analysis, decision making, and some other areas involving the manipulation of imprecise data.

2. **Preliminaries**

Basic definitions on fuzzy sets and fuzzy finite automata can be found in [19] and [14]. In this section, we have given some definitions.

Definition 2.1. Let Σ be a finite alphabet set and $f_{\tilde{L}}(w) : \Sigma^* \to M$ a function, where M is a set of real numbers in $[0,1]$. Then the set $\tilde{L} = \{(w, f_{\tilde{L}}(w)) \mid w \in \Sigma^*\}$ is called a fuzzy language (FL) [15] over Σ and $f_{\tilde{L}}(w)$ the membership function of $\tilde{L}$.

Definition 2.2. Let '$\tilde{L}$' be a fuzzy language over Σ, the finite alphabet set with $f_{\tilde{L}}(w) : \Sigma^* \to M$ as its membership function. Then, '$\tilde{L}$' is called a fuzzy regular language (FRL) [15] if;

(i) the set $\{m \in M \mid S_{\tilde{L}}(m) \neq \emptyset\}$ is finite, and

(ii) for each $m \in M$, the string $S_{\tilde{L}}(m)$ is regular,

where $S_{\tilde{L}}(m) = \{w \in \Sigma^* \mid f_{\tilde{L}}(w) = m\}$ [15] and a string is regular, if it is recognized by a finite automaton [9].

Definition 2.3. Let Σ be a set of finite alphabet and $f_{\tilde{L}}(w) : \Sigma^* \to M$, $g_{\tilde{L}}(w) : \Sigma^* \to N$ are the functions, where M and N are the finite set of real numbers in [0, 1]. Then we call the set, $\tilde{L} = \{(w, f_{\tilde{L}}(w), g_{\tilde{L}}(w)) \mid w \in \Sigma^*\}$ an intuitionistic fuzzy language (IFL) [17] over Σ.
Here, $f_{\tilde{L}}(w), g_{\tilde{L}}(w)$ represents respectively, the membership and nonmembership functions of $\tilde{L}$ and for any $w \in \Sigma^*$, $0 \leq f_{\tilde{L}}(w) + g_{\tilde{L}}(w) \leq 1$.

Definition 2.4. Let '$\tilde{L}$' be an IFL over Σ, the finite alphabet set with $f_{\tilde{L}}(w)$ and $g_{\tilde{L}}(w)$ as its membership and nonmembership functions. We call '$\tilde{L}$' an intuitionistic fuzzy regular language (IFRL) [17] if,

(i) the sets $\{m \in M \mid S_{\tilde{L}}(m) \neq \emptyset\}$ and $\{l \in N \mid S_{\tilde{L}}(l) \neq \emptyset\}$ are finite, and

(ii) for each $m \in M$ the string $S_{\tilde{L}}(m)$ and for each $l \in N$ the string $S_{\tilde{L}}(l)$ are regular,

where $S_{\tilde{L}}(m) = \{w \in \Sigma^* \mid f_{\tilde{L}}(w) = m\}$ and $S_{\tilde{L}}(l) = \{w \in \Sigma^* \mid g_{\tilde{L}}(w) = l\}$ [17].

Definition 2.5. Let Σ be a finite alphabet set. Then we call the set, $\tilde{L} = \{(w, [f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w)]) \mid w \in \Sigma^*\}$ an interval-valued fuzzy language (IVFL) [18], where $f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w) : \Sigma^* \to [0,1]$ represents the lower and the upper membership functions of $\tilde{L}$ respectively.
Here, for any $w \in \Sigma^*, 0 \leq f^L_{\tilde{L}}(w) \leq f^U_{\tilde{L}}(w) \leq 1$ and $0 \leq f^L_{\tilde{L}}(w) + (1 - f^U_{\tilde{L}}(w)) \leq 1$.

In short, $\tilde{L} = \{(w, f_{\tilde{L}}(w)) \mid w \in \Sigma^*\}$, where $f_{\tilde{L}}(w) = [f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w)]\ \forall w \in \Sigma^*$.

Definition 2.6. Let '$\tilde{L}$' be an IVFL over Σ, the finite alphabet set, and $f_{\tilde{L}}(w)$ the membership function of '$\tilde{L}$'. Then we call, '$\tilde{L}$' an interval-valued fuzzy regular language (IVFRL) [18] if;

(i) the set $\{[m,n] \in I[0,1] \mid S_{\tilde{L}}[m,n] \neq \emptyset\}$ is finite, and

(ii) for each $[m,n] \in I[0,1]$ the string $S_{\tilde{L}}[m,n]$ is regular,

where $S_{\tilde{L}}[m,n] = \{w \in \Sigma^* \mid f_{\tilde{L}}(w) = [m,n]\}$ [18].

Definition 2.7. Let Σ be a finite alphabet set. Then we call the set, $\tilde{L} = \{(w, [t_{\tilde{L}}(w), 1 - f_{\tilde{L}}(w)]) \mid w \in \Sigma^*\}$ a vague language (VL) [6] over Σ. Here, $t_{\tilde{L}}(w), f_{\tilde{L}}(w) : \Sigma^* \to [0,1]$ represents respectively, the truth membership and the false membership functions of $\tilde{L}$, such that $0 \leq t_{\tilde{L}}(w) \leq 1 - f_{\tilde{L}}(w) \leq 1$ or
$0 \leq t_{\tilde{L}}(w) + f_{\tilde{L}}(w) \leq 1$.

Definition 2.8. Let '$\tilde{L}$' be a VL over Σ, the finite alphabet set with $t_{\tilde{L}}(w) : \Sigma^* \to M$, $f_{\tilde{L}}(w) : \Sigma^* \to N$ as its truth and false membership functions respectively. Then we call, '$\tilde{L}$' a vague regular language (VRL) [6] if,

(i) the sets $\{m \in M \mid S_{\tilde{L}}(m) \neq \emptyset\}$ and $\{n \in N \mid S_{\tilde{L}}(1-n) \neq \emptyset\}$ are finite, and

(ii) for each $m \in M$ and $n \in N$ the strings $S_{\tilde{L}}(m)$ and $S_{\tilde{L}}(1-n)$ are regular,

where $S_{\tilde{L}}(m) = \{w \in \Sigma^* \mid t_{\tilde{L}}(w) = m\}$ and $S_{\tilde{L}}(1-n) = \{w \in \Sigma^* \mid 1 - f_{\tilde{L}}(w) = 1 - n\}$ [6].

Definition 2.9. A nondeterministic finite automaton with vague (final) states (NDFA-VS) [6] '$\tilde{A}$' is a 7-tuple $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{T}_{F_{\tilde{A}}}, \tilde{F}_{F_{\tilde{A}}})$, where Q is the finite set of states, Σ is the finite set of input alphabets, δ, γ: $Q \times \Sigma :\to 2^Q$ are the state transition functions i.e., $\delta(p,a) = q$ and $\gamma(p,a) = q$ for $p \in Q, q \in 2^Q$ and $a \in \Sigma$, q_0 is the vague starting state and $\tilde{T}_{F_{\tilde{A}}}$, $\tilde{F}_{F_{\tilde{A}}}$: $Q \to [0,1]$ are respectively, the truth and false membership functions of vague (final) state set.

Define,

$td_{\tilde{A}}(x) = \max\{\tilde{T}_{F_{\tilde{A}}}(q) \mid (q_0, x, q) \in \delta^*\}$ and

$fd_{\tilde{A}}(x) = \min\{\tilde{F}_{F_{\tilde{A}}}(q) \mid (q_0, x, q) \in \gamma^*\}$ or $fd_{\tilde{A}}(x) = \max\{1 - \tilde{F}_{F_{\tilde{A}}}(q) \mid (q_0, x, q) \in \gamma^*\}$, where $\delta^*, \gamma^* : Q \times \Sigma^* \to 2^Q$ are respectively, the reflexive and transitive closure of δ and γ.
The string 'x' is accepted by '$\tilde{A}$' with the truth degree $td_{\tilde{A}}(x)$ and the false degree $fd_{\tilde{A}}(x)$ with the condition $0 \leq td_{\tilde{A}}(x) + fd_{\tilde{A}}(x) \leq 1$.

The vague regular language accepted by '$\tilde{A}$' is denoted by $\tilde{L}(\tilde{A})$ and is given by the set, $\tilde{L}(\tilde{A}) = \{(x, [td_{\tilde{A}}(x), 1 - fd_{\tilde{A}}(x)]) \mid x \in \Sigma^*\}$.

Definition 2.10. A deterministic finite automaton with vague (final) states (DFA-VS) [6] $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{T}_{F_{\tilde{A}}}, \tilde{F}_{F_{\tilde{A}}})$ is a NDFA-VS with δ, γ: $Q \times \Sigma \to Q$ being functions instead of a relation.

For each $x \in \Sigma^*$, $td_{\tilde{A}}(x) = \tilde{T}_{F_{\tilde{A}}}(q)$, where $q = \delta^*(q_0, x)$ and
$fd_{\tilde{A}}(x) = \tilde{F}_{F_{\tilde{A}}}(q)$, where $q = \gamma^*(q_0, x)$.
Define, $td_{\tilde{A}}(x) = 0$ and $fd_{\tilde{A}}(x) = 1$ if $\delta^*(q_0, x)$ and $\gamma^*(q_0, x)$ are not defined (i.e., there is no transition for a string x from state q_0).

Note: Deterministic and nondeterministic finite automata with vague (final) states are called vague (final) states automaton.

Definition 2.11. A nondeterministic finite automaton with intuitionistic fuzzy (final) states (NDFA-IFS) [5] '$\tilde{A}$' is a 7-tuple $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$, where Q is the finite set of states, Σ is the finite set of input alphabets, δ, γ: $Q \times \Sigma :\to 2^Q$ are the state transition functions i.e., $\delta(p, a) = q$ and $\gamma(p, a) = q$ for $p \in Q, q \in 2^Q$ and $a \in \Sigma$, q_0 is the intuitionistic fuzzy starting state and $\tilde{F}_{1\tilde{A}}$, $\tilde{F}_{2\tilde{A}}$: $Q \to [0, 1]$ represents the membership and the nonmembership functions of intuitionistic fuzzy (final) state set respectively.

Define,
$d_{\tilde{A}}(x) = \max\{\tilde{F}_{1\tilde{A}}(q) \mid (q_0, x, q) \in \delta^*\}$ and
$n_{\tilde{A}}(x) = \min\{\tilde{F}_{2\tilde{A}}(q) \mid (q_0, x, q) \in \gamma^*\}$, where $\delta^*, \gamma^* : Q \times \Sigma^* \to 2^Q$ are the reflexive and transitive closure of δ and γ respectively.
The string 'x' is accepted by '$\tilde{A}$' with the degree $d_{\tilde{A}}(x)$ and the nondegree $n_{\tilde{A}}(x)$ such that $0 \leq d_{\tilde{A}}(x) + n_{\tilde{A}}(x) \leq 1$.
The intuitionistic fuzzy regular language accepted by '$\tilde{A}$' is denoted by $\tilde{L}(\tilde{A})$ and is given by the set, $\tilde{L}(\tilde{A}) = \{(x, d_{\tilde{A}}(x), n_{\tilde{A}}(x)) \mid x \in \Sigma^*\}$.

Definition 2.12. A deterministic finite automaton with intuitionistic fuzzy (final) states (DFA-IFS) [5] $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ is a NDFA-IFS with
δ, γ: $Q \times \Sigma \to Q$ being functions instead of a relation.

For each $x \in \Sigma^*$, $d_{\tilde{A}}(x) = \tilde{F}_{1\tilde{A}}(q)$, where $q = \delta^*(q_0, x)$ and
$n_{\tilde{A}}(x) = \tilde{F}_{2\tilde{A}}(q)$, where $q = \gamma^*(q_0, x)$.
Define, $d_{\tilde{A}}(x) = 0$ and $n_{\tilde{A}}(x) = 1$ if $\delta^*(q_0, x)$ and $\gamma^*(q_0, x)$ are not defined (i.e., there is no transition for a string x from state q_0).

Note: Deterministic and nondeterministic finite automata with intuitionistic fuzzy (final) states are called intuitionistic fuzzy (final) states automaton.

3. Relation Between IFL, IVFL and VL

Intuitionistic fuzzy sets, interval-valued fuzzy sets and vague sets are the extensions of fuzzy set. Atanassov and Gargov [3] shown that, interval-valued fuzzy sets can be express in the form of intuitionistic fuzzy sets and Bustince and Burillo [4] shown that vague sets are equivalent to intuitionistic fuzzy sets. In this section, an attempt has been made to compare three models that extend fuzzy language theory: intuitionistic fuzzy language, interval-valued fuzzy language and vague language theory. Our exposition recalls the concept of their membership values resulting in some relations among them. With the help of this, the property of one language can be used to study the property of another.

The difference between IFL and IVFL is due to the definition of their membership values. In IFL, we have $(f_{\tilde{L}}(w), g_{\tilde{L}}(w))$ as the membership value of a string 'w', where each of $f_{\tilde{L}}(w)$ and $g_{\tilde{L}}(w)$ represents a value of 'w' in [0, 1]. These are respectively, the membership and the nonmembership values of 'w', with the condition that $0 \leq f_{\tilde{L}}(w) + g_{\tilde{L}}(w) \leq 1$. Whereas, in IVFL the membership value of a string 'w' is given by $[f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w)]$, where each of $f^L_{\tilde{L}}(w)$ and $f^U_{\tilde{L}}(w)$ represents a value in [0, 1]. These represents respectively, the lower and the upper membership values of 'w', with the condition that $0 \leq f^L_{\tilde{L}}(w) + (1 - f^U_{\tilde{L}}(w)) \leq 1$. It has been observed that the semantics of the membership value $(f_{\tilde{L}}(w))$ of a string 'w' in IFL is the same as lower membership value $(f^L_{\tilde{L}}(w))$ of the string 'w' in IVFL and the nonmembership value $(g_{\tilde{L}}(w))$ of the string 'w' in IFL is the same as $(1 - f^U_{\tilde{L}}(w))$ (i.e., complement of the upper membership value) of the string 'w' in IVFL.

Also, the membership value plays an important role while differentiating IFL and VL. The membership value of a string 'w' in IFL is explained above. The membership value of a string 'w' in VL is given by $[t_{\tilde{L}}(w), 1 - f_{\tilde{L}}(w)]$, where each of $t_{\tilde{L}}(w)$ and $f_{\tilde{L}}(w)$ represents a value in [0, 1]. These are respectively, the truth membership and the false membership values of 'w', with the condition that $0 \leq t_{\tilde{L}}(w) + f_{\tilde{L}}(w) \leq 1$. It has been observed that the semantics of the membership value $(f_{\tilde{L}}(w))$ of 'w' in IFL is the same as that of the truth membership value $(t_{\tilde{L}}(w))$ of 'w' in VL and the nonmembership value $(g_{\tilde{L}}(w))$ of 'w' in IFL is the same as that of the false membership value $(f_{\tilde{L}}(w))$ of 'w' in VL.

From the above discussion, one can obtain an equality relation between the membership values of the string 'w' in aforementioned languages (in sequence IFL, IVFL and VL) as follows:

(i) $f_{\tilde{L}}(w) = f^L_{\tilde{L}}(w) = t_{\tilde{L}}(w)$ and

(ii) $g_{\tilde{L}}(w) = 1 - f^U_{\tilde{L}}(w) = f_{\tilde{L}}(w)$.

4. Myhill-Nerode Theorem for Vague Regular Language

One classical problem in the theory of automata is equivalence, reduction and minimization of finite state automata. Myhill-Nerode theory is a branch of the algebraic theory of languages and automata in which formal languages and deterministic automata are studied through right invariant equivalence classes. This theorem has also been extended to FRL. In our study we have further extended Myhill-Nerode theory to VRL and IFRL and have provided an efficient algorithm for minimizing DFA-VS and DFA-IFS. It provides necessary and sufficient conditions for VL and IFL to be regular in terms of equivalence classes. In particular, right invariant equivalence classes have shown oneself to be very useful in the proof of existence and construction of the minimal DFA-VS and DFA-IFS recognizing VRL and IFRL, respectively.

Theorem 4.1. *The following three statements are equivalent to one another:*

(i) Some finite automaton with vague (final) states can accept a vague regular language $\tilde{L}$ over Σ.
(ii) $\tilde{L}$ is the union of some equivalence classes of a right invariant equivalence relation of finite index.
(iii) Let the relation $R_{\tilde{L}} \subseteq \Sigma^* \times \Sigma^*$ be defined as $xR_{\tilde{L}}y$ iff $\forall\, z \in \Sigma^*$, $t_{\tilde{L}}(xz) = t_{\tilde{L}}(yz)$ and $f_{\tilde{L}}(xz) = f_{\tilde{L}}(yz)$, then $R_{\tilde{L}}$ is an equivalence relation of finite index.

Proof. $(i) \Rightarrow (ii)$ Let $\tilde{L}$ be a vague regular language over Σ the finite alphabet set. Assume that $\tilde{L}$ is accepted by some DFA-VS $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{T}_{F_{\tilde{A}}}, \tilde{F}_{F_{\tilde{A}}})$. Let $R_{\tilde{A}}$ be the equivalence relation $xR_{\tilde{A}}y$ iff $\delta(q_0, x) = \delta(q_0, y)$ and $\gamma(q_0, x) = \gamma(q_0, y)$. $R_{\tilde{A}}$ is right invariant since, for any z, $\delta(q_0, xz) = \delta(q_0, yz)$, if $\delta(q_0, x) = \delta(q_0, y)$ and $\gamma(q_0, xz) = \gamma(q_0, yz)$, if $\gamma(q_0, x) = \gamma(q_0, y)$. Then the index of $R_{\tilde{A}}$ is finite, since the index is at most the number of states in Q. Furthermore, $\tilde{L}$ is the union of those equivalence classes having a string 'x' such that $\delta(q_0, x)$ and $\gamma(q_0, x)$ is in $\tilde{T}_{F_{\tilde{A}}}$ and $\tilde{F}_{F_{\tilde{A}}}$ respectively (*i.e.,* the equivalence classes corresponding to the final states).

$(ii) \Rightarrow (iii)$ We show that any equivalence relation 'E' satisfying (ii) is a refinement of $R_{\tilde{L}}$; *i.e.,* some equivalence class of $R_{\tilde{L}}$ will be the superset of every equivalence class 'E'. Thus, the index of $R_{\tilde{L}}$ cannot be greater than the index of 'E' and so is finite. Assume that 'xEy'. For each $z \in \Sigma^*$, '$xzEyz$' and thus $\tilde{L}(xz) = \tilde{L}(yz)$ (since 'E' is right invariant). Hence, $xR_{\tilde{L}}y$. We conclude that each equivalence class of 'E' is the subset of some equivalence class of $R_{\tilde{L}}$.

$(iii) \Rightarrow (i)$ To show that $R_{\tilde{L}}$ is right invariant, suppose $xR_{\tilde{L}}y$, and let $w \in \Sigma^*$, we must prove that $xwR_{\tilde{L}}yw$; *i.e.,* for any z, $\tilde{L}(xwz) = \tilde{L}(ywz)$. Since $xR_{\tilde{L}}y$, for any v, $\tilde{L}(xv) = \tilde{L}(yv)$ (by the definition of $R_{\tilde{L}}$). Consider '$v = wz$' to prove $R_{\tilde{L}}$ is right invariant.

Now we present the minimized DFA-VS by constructing equivalence classes of $R_{\tilde{L}}$: Let Q' be the finite set of equivalence classes of $R_{\tilde{L}}$ and $[x] \in Q'$ containing x. Define, $\delta'([x], a) = [xa]$ and $\gamma'([x], a) = [xa]$. This definition is consistent as $R_{\tilde{L}}$ is right invariant. If we choose 'y' instead of 'x' from [x], we will have $\delta'([x], a) = [ya]$ and $\gamma'([x], a) = [ya]$. But $xR_{\tilde{L}}y$, so $\tilde{L}(xz) = \tilde{L}(yz)$. In particular, if $z = az'$, $\tilde{L}(xaz') = \tilde{L}(yaz')$, so $xaR_{\tilde{L}}ya$ and [xa]=[ya].

Let $q_0' = [\epsilon]$, $\tilde{T}'_{F_{\tilde{A}}} = \{[x] \mid x \in \tilde{L}\}$ and $\tilde{F}'_{F_{\tilde{A}}} = \{[x] \mid x \in \tilde{L}\}$. The finite automaton $\tilde{A}' = (Q', \Sigma, \delta', \gamma', q_0', \tilde{T}'_{F_{\tilde{A}'}}, \tilde{F}'_{F_{\tilde{A}'}})$ accepts $\tilde{L}$, since $\delta'(q_0', x) = \delta'([\epsilon], x) = [x]$ and $\gamma'(q_0', x) = \gamma'([\epsilon], x) = [x]$. Thus $\tilde{L}(\tilde{A}') = \tilde{L}(\tilde{A})$. □

Algorithm 4.2. *Algorithm for minimizing deterministic finite automata with vague (final) states (DFA-VS)*

Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{T}_{F_{\tilde{A}}}, \tilde{F}_{F_{\tilde{A}}})$ be a DFA-VS. Assume that $Q = \{q_0, q_1, ..., q_n\}$, $n \geq 0$ and let $P = \{(q_i, q_j) \mid q_i, q_j \in Q \ \ and \ \ 0 \leq i < j \leq n\}$.

begin

Step 1: **for** each pair $(q_i, q_j) \in P$, and $\tilde{T}_{F_{\tilde{A}}}(q_i) \neq \tilde{T}_{F_{\tilde{A}}}(q_j)$ or $\tilde{F}_{F_{\tilde{A}}}(q_i) \neq \tilde{F}_{F_{\tilde{A}}}(q_j)$ do mark (q_i, q_j);

Step 2: **for** each unmarked pair $(q_i, q_j) \in P$ do

if for some $x \in \Sigma$, $(\delta(q_i, x), \delta(q_j, x))$ and $(\gamma(q_i, x), \gamma(q_j, x))$ is marked then

Step 2.1: mark (q_i, q_j);

Step 2.2: recursively mark all unmarked pairs on the list of (q_i, q_j) and on the list of other pairs that are marked at this step.

else

Step 2.3: **for** all input symbols 'x' do put (q_i, q_j) on the list for $(\delta(q_i, x), \delta(q_j, x))$ and $(\gamma(q_i, x), \gamma(q_j, x))$ unless $\delta(q_i, x) = \delta(q_j, x)$ and $\gamma(q_i, x) = \gamma(q_j, x)$.

Step 3: Equivalence classes of Q are constructed as follows;

For i = 0 to n - 1 do

For j = i + 1 to n do

if (q_i, q_j) is unmarked, q_j is in $[q_i]$, the equivalence class containing q_i.

Step 4: Define a minimum DFA-VS $\tilde{A}' = (Q', \Sigma, \delta', \gamma', q_0', \tilde{T}'_{F_{\tilde{A}'}}, \tilde{F}'_{F_{\tilde{A}'}})$ as follows;

$Q' = \{[q_i] \mid q_i \in Q\}$, $\delta'([q_i], a) = [\delta(q_i, a)]$, $\gamma'([q_i], a) = [\gamma(q_i, a)]$, $q_0' = [q_0]$, $\tilde{T}'_{F_{\tilde{A}'}}([q_i]) = \tilde{T}_{F_{\tilde{A}}}(q_i)$ and $\tilde{F}'_{F_{\tilde{A}'}}([q_i]) = \tilde{F}_{F_{\tilde{A}}}(q_i)$.

end.

Example 4.3. Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{T}_{F_{\tilde{A}}}, \tilde{F}_{F_{\tilde{A}}})$ be a DFA-VS (Figure 1). Here, $Q = \{a, b, c, d, e, f\}$, $\Sigma = \{0, 1\}$, $q_0 = \{a\}$ the vague starting state with truth membership value $\tilde{T}_{F_{\tilde{A}}}(a) = 0.4$ and false membership value $\tilde{F}_{F_{\tilde{A}}}(a) = 0.5$. δ, $\gamma : Q \times \Sigma \to Q$ are the transition functions given as $\delta(a, 0) = \gamma(a, 0) = c$, $\delta(a, 1) = \gamma(a, 1) = b$, $\delta(b, 0) = \gamma(b, 0) = d$, $\delta(b, 1) = \gamma(b, 1) = a$, $\delta(c, 0) = \gamma(c, 0) = f$, $\delta(c, 1) = \gamma(c, 1) = e$, $\delta(d, 0) = \gamma(d, 0) = f$, $\delta(d, 1) = \gamma(d, 1) = e$, $\delta(e, 0) = \gamma(e, 0) = f$, $\delta(e, 1) = \gamma(e, 1) = e$, $\delta(f, 0) = \gamma(f, 0) = f$, $\delta(f, 1) = \gamma(f, 1) = f$, and $\tilde{T}_{F_{\tilde{A}}}(b) = 0.4$, $\tilde{F}_{F_{\tilde{A}}}(b) = 0.6$, $\tilde{T}_{F_{\tilde{A}}}(c) = 0.6$, $\tilde{F}_{F_{\tilde{A}}}(c) = 0.7$, $\tilde{T}_{F_{\tilde{A}}}(d) = 0.6$, $\tilde{F}_{F_{\tilde{A}}}(d) = 0.7$, $\tilde{T}_{F_{\tilde{A}}}(e) = 0.7$, $\tilde{F}_{F_{\tilde{A}}}(e) = 0.9$, and $\tilde{T}_{F_{\tilde{A}}}(f) = 0.2$, $\tilde{F}_{F_{\tilde{A}}}(f) = 0.3$ shows the truth and false membership values of the states {b}, {c}, {d}, {e}, and {f} respectively.

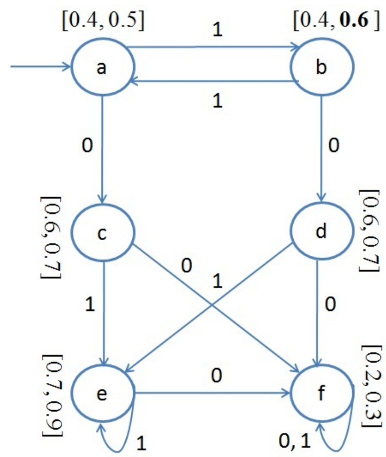

FIGURE 1. DFA-VS

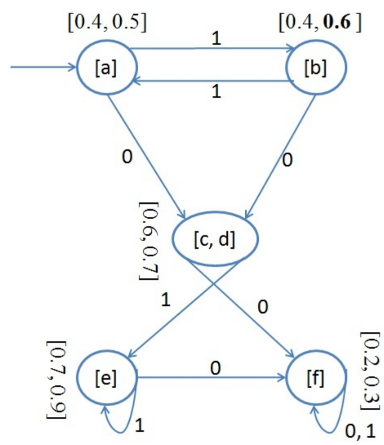

FIGURE 2. Minimized DFA-VS of Figure 1

Above DFA-VS (Figure 1) and its minimized DFA-VS (Figure 2) will accept the vague regular language;
$\tilde{L} = \{\mathbf{1(11)}^*/[\mathbf{0.4}, \mathbf{0.6}],\ (11)^*/[0.4, 0.5],\ 1^*0/[0.6, 0.7],\ 1^*01^+/[0.7, 0.9],\ 1^*00(0+1)^*/[0.2, 0.3],\ 1^*01^+0(0+1)^*/[0.2, 0.3]\}$.

Similarly, we can prove the Myhill-Nerode theorem for intuitionistic fuzzy regular language. An algorithm for minimizing DFA-IFS is given below.

Algorithm 4.4. *Algorithm for Minimizing Deterministic Finite Automata with Intuitionistic fuzzy (final) States (DFA-IFS)*

Let $\tilde{B} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{F}_{1\tilde{B}}, \tilde{F}_{2\tilde{B}})$ be a DFA-IFS [5]. Assume that $Q = \{q_0, q_1, ..., q_n\},\ \ n \geq 0$ and let $P = \{(q_i, q_j) \mid q_i, q_j \in Q \ \ and \ \ 0 \leq i < j \leq n\}$.

begin

Step 1: **for** each pair $(q_i, q_j) \in P$, and $\tilde{F}_{1\tilde{B}}(q_i) \neq \tilde{F}_{1\tilde{B}}(q_j)$ or $\tilde{F}_{2\tilde{B}}(q_i) \neq \tilde{F}_{2\tilde{B}}(q_j)$ do mark (q_i, q_j);

Step 2: **for** each unmarked pair $(q_i, q_j) \in P$ do

if for some $x \in \Sigma$, $(\delta(q_i, x), \delta(q_j, x))$ and $(\gamma(q_i, x), \gamma(q_j, x))$ is marked then

Step 2.1: mark (q_i, q_j);

Step 2.2: recursively mark all unmarked pairs on the list of (q_i, q_j) and on the list of other pairs that are marked at this step.

else

Step 2.3: **for** all input symbols 'x' do put (q_i, q_j) on the list for $(\delta(q_i, x), \delta(q_j, x))$ and $(\gamma(q_i, x), \gamma(q_j, x))$ unless $\delta(q_i, x) = \delta(q_j, x)$ and $\gamma(q_i, x) = \gamma(q_j, x)$.

Step 3: Equivalence classes of Q are constructed as follows;

For i = 0 to n - 1 do

For j = i + 1 to n do

if (q_i, q_j) is unmarked, q_j is in $[q_i]$, the equivalence class containing q_i.

Step 4: Define a minimum DFA-IFS $\tilde{B}' = (Q', \Sigma, \delta', \gamma', q_0', \tilde{F}'_{1\tilde{B}'}, \tilde{F}'_{2\tilde{B}'})$ as follows;

$Q' = \{[q_i] \mid q_i \in Q\}$, $\delta'([q_i], a) = [\delta(q_i, a)]$, $\gamma'([q_i], a) = [\gamma(q_i, a)]$,

$q_0' = [q_0]$, $\tilde{F}'_{1\tilde{B}'}([q_i]) = \tilde{F}_{1\tilde{B}}(q_i)$ and $\tilde{F}'_{2\tilde{B}'}([q_i]) = \tilde{F}_{2\tilde{B}}(q_i)$.

end.

If intuitionistic fuzzy sets are reduced to fuzzy sets, we will consider only membership value. In that case DFA-IFS becomes deterministic finite automaton with fuzzy (final) states (DFA-FS). This DFA-FS may reduce further depending on the state transition and membership of each states (where strings with membership 0 in DFA-FS will not be considered as explained in example 4.2). For reducing DFA-FS we apply the algorithm given in [15]. Again, if fuzzy set is reduced to crisp set, then we will consider state with membership value zero in DFA-FS as non-final state in deterministic finite automaton (DFA) and all other states as final states in DFA. Thus DFA-FS becomes DFA. Depending on the state transition, DFA may get further reduced. For reducing automata, we will apply algorithm given in [9].

Further if intuitionistic fuzzy sets reduced to fuzzy sets, intuitionistic fuzzy regular language reduces to fuzzy refular language, where each string contains some membership value in [0, 1] and it can be recognized by DFA-FS and NDFA-FS [15]. The state reduction of DFA-FS will depend on membrship value of the state and state transition. Again, if fuzzy sets are reduced to crisp sets, fuzzy regular language becomes regular language. This regular language is accepted by a finite automaton and same can be reduced to its minimized form if possible [9].

Example 4.5. Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ be a DFA-IFS (Figure 3). Here, $Q = \{p, q, r, s, t, u\}$, $\Sigma = \{a, b\}$, $q_0 = \{p\}$ the intuitionistic fuzzy starting state with membership value $\tilde{F}_{1\tilde{A}}(p) = 0.2$ and nonmembership value $\tilde{F}_{2\tilde{A}}(p) = 0.4$. δ,

$\gamma : Q \times \Sigma \to Q$ are the transition functions given as $\delta(p,a) = \gamma(p,a) = q$, $\delta(p,b) = \gamma(p,b) = t$, $\delta(q,a) = \gamma(q,a) = s$, $\delta(q,b) = \gamma(q,b) = r$, $\delta(r,a) = \gamma(r,a) = t$, $\delta(r,b) = \gamma(r,b) = u$, $\delta(s,a) = \gamma(s,a) = u$, $\delta(s,b) = \gamma(s,b) = r$, $\delta(t,a) = \gamma(t,a) = t$, $\delta(t,b) = \gamma(t,b) = t$, $\delta(u,a) = \gamma(u,a) = u$, $\delta(u,b) = \gamma(u,b) = u$, and $\tilde{F}_{1\tilde{A}}(q) = 0.3$, $\tilde{F}_{2\tilde{A}}(q) = 0.5$, $\tilde{F}_{1\tilde{A}}(r) = 0.2$, $\tilde{F}_{2\tilde{A}}(r) = 0.4$, $\tilde{F}_{1\tilde{A}}(s) = 0.3$, $\tilde{F}_{2\tilde{A}}(s) = 0.6$, $\tilde{F}_{1\tilde{A}}(t) = 0$, $\tilde{F}_{2\tilde{A}}(t) = 1$, and $\tilde{F}_{1\tilde{A}}(u) = 0$, $\tilde{F}_{2\tilde{A}}(u) = 1$ shows the membership and nonmembership value of the states {q}, {r}, {s}, {t}, and {u} respectively.

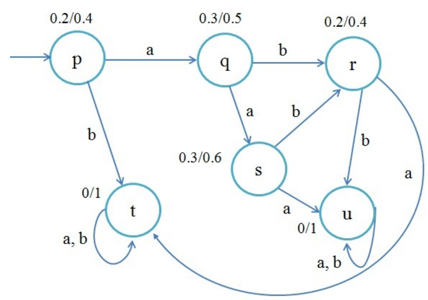

Figure 3. DFA-IFS

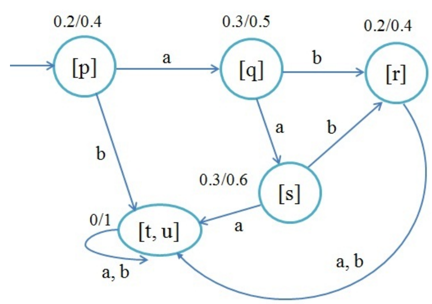

Figure 4. Minimized DFA-IFS of Figure 3

Above DFA-IFS (Figure 3) and its minimized DFA-IFS (Figure 4) will accept the intuitionistic fuzzy regular language;

$\tilde{L} = \{\epsilon/0.2/0.4, a/0.3/0.5, aa/0.3/0.6, ab, aab/0.2/0.4\}$.

If we change this DFA-IFS (Figure 3) to DFA-FS, the number of states of reduced DFA-FS [15] is the same as the number of states of reduced DFA-IFS. Again, if we change DFA-FS to DFA, the number of states of reduced DFA [9] will be the same as the number of states of reduced DFA-FS (here, (in DFA) we consider states with membership zero in DFA-FS as non final states).

Example 4.6. Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ be a DFA-IFS (Figure 5). Here, $Q = \{a, b, c, d, e\}$, $\Sigma = \{0, 1\}$, $q_0 = \{a\}$ the intuitionistic fuzzy starting state with membership value $\tilde{F}_{1\tilde{A}}(a) = 0.5$ and nonmembership value $\tilde{F}_{2\tilde{A}}(a) = 0.5$. δ, $\gamma : Q \times \Sigma \to Q$ are the transition functions given as $\delta(a, 0) = \gamma(a, 0) = b$, $\delta(a, 1) = \gamma(a, 1) = d$, $\delta(b, 0) = \gamma(b, 0) = c$, $\delta(b, 1) = \gamma(b, 1) = e$, $\delta(c, 0) = \gamma(c, 0) = b$, $\delta(c, 1) = \gamma(c, 1) = e$, $\delta(d, 0) = \gamma(d, 0) = c$, $\delta(d, 1) = \gamma(d, 1) = e$, $\delta(e, 0) = \gamma(e, 0) = e$, $\delta(e, 1) = \gamma(e, 1) = e$, and $\tilde{F}_{1\tilde{A}}(b) = 0.6$, $\tilde{F}_{2\tilde{A}}(b) = 0.3$, $\tilde{F}_{1\tilde{A}}(c) = 0.6$, $\tilde{F}_{2\tilde{A}}(c) = 0.2$, $\tilde{F}_{1\tilde{A}}(d) = 0.6$, $\tilde{F}_{2\tilde{A}}(d) = 0.3$, and $\tilde{F}_{1\tilde{A}}(e) = 0$, $\tilde{F}_{2\tilde{A}}(e) = 1$ shows the membership and nonmembership value of the states {b}, {c}, {d}, and {e} respectively.

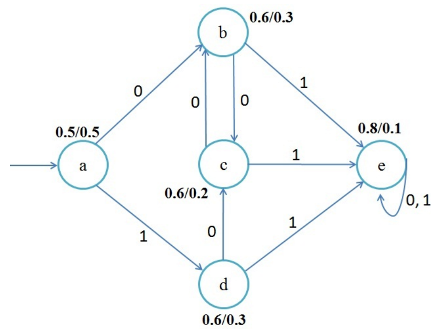

FIGURE 5. DFA-IFS

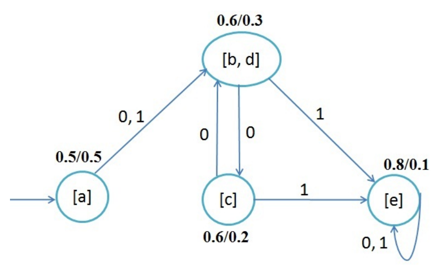

FIGURE 6. Minimized DFA-IFS of Figure 5

Above DFA-IFS (Figure 5) and its minimized DFA-IFS (Figure 6) will accept the intuitionistic fuzzy regular language;

$\tilde{L} = \{\epsilon/0.5/0.5,\ 0(00)^*/0.6/0.3,\ 0(00)^*0/0.6/0.2,\ 1(00)^*/0.6/0.3,\ 1(00)^*0/0.6/0.2,\ 0(00)^*1(0+1)^*/0.8/0.1,\ 1(00)^*1(0+1)^*/0.8/0.1,\ 0(00)^*01(0+1)^*/0.8/0.1,\ 1(00)^*01(0+1)^*/0.8/0.1\}$.

DFA-IFS (Figure 5) is changed to DFA-FS (Figure 7) and its minimized DFA-FS is given in Figure 8 accepting the fuzzy regular language

$\tilde{L} = \{\epsilon/0.5, 0^+, 10^*/0.6, 0^+1(0+1)^*, 10^*1(0+1)^*/0.8\}$ [13].

DFA-FS (Figure 7) is changed to DFA (Figure 9). Its minimized DFA is given in Figure 10 and both of these will accept the regular language; $\tilde{L} = \{0+1\}^*$ [9].

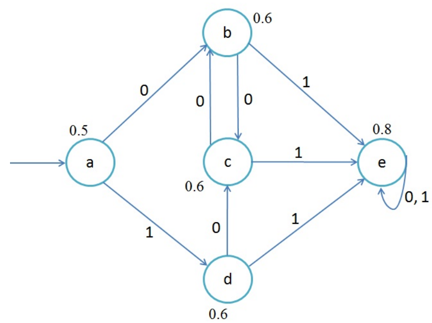

Figure 7. DFA-FS

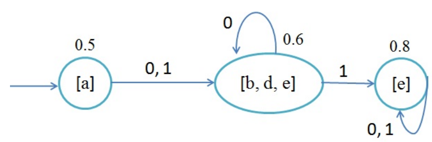

Figure 8. Minimized DFA-FS of Figure 7

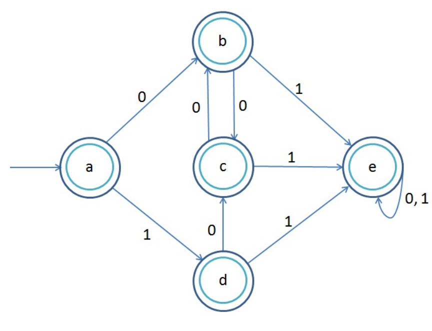

Figure 9. DFA

5. **Conclusion**

The extensive research being done on intuitionistic fuzzy sets, interval-valued fuzzy sets and vague sets (a survey [16] lists over 400 publications in the domain of intuitionistic fuzzy set theory alone, and the number is still growing fast) shows a mounting interest in these models. This paper has attempted to mend the situation by obtaining a relation between the membership values of IFL, IVFL and VL. It discusses the extended Myhill-Nerode theorem in the framework of VRL and IFRL also, it explains the method of minimizing DFA-VS and DFA-IFS through an algorithm. The theory of VL and IFL may prove to be of relevance in the construction of better models for natural languages. These may contribute to a better

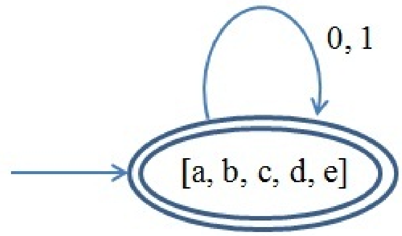

Figure 10. Minimized DFA of Figure 9

understanding of the role of vague (final) states automaton or intuitionistic fuzzy (final) states automaton in lexical analysis, decision making, pattern recognition, learning systems and other processes involving the manipulation of imprecise data.

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ON THE DIAGRAM OF ONE TYPE MODAL OPERATORS ON INTUITIONISTIC FUZZY SETS: LAST EXPANDING WITH $Z^{\omega,\theta}_{\alpha,\beta}$

G. ÇUVALCIOĞLU

Abstract. Intuitionistic Fuzzy Modal Operator was defined by Atanassov in [3] in 1999. In 2001, [4], he introduced the generalization of these modal operators. After this study, in 2004, Dencheva [14] defined second extension of these operators. In 2006, the third extension of these was defined in [6] by Atanassov. In 2007,[11], the author introduced a new operator over Intuitionistic Fuzzy Sets which is a generalization of Atanassov's and Dencheva's operators. At the same year, Atanassov defined an operator which is an extension of all the operators defined until 2007. The diagram of One Type Modal Operators on Intuitionistic Fuzzy Sets was introduced first in 2007 by Atanassov [10]. In 2008, Atanassov defined the most general operator and in 2010 the author expanded the diagram of One Type Modal Operators on Intuitionistic Fuzzy Sets with the operator $Z^{\omega}_{\alpha,\beta}$. Some relationships among these operators were studied by several researchers[5]-[8] [11], [13], [14]- [19]. The aim of this paper is to expand the diagram of one type modal operators over intuitionistic fuzzy sets . For this purpose, we defined a new modal oparator $Z^{\omega,\theta}_{\alpha,\beta}$ over intuitionistic fuzzy sets. It is shown that this oparator is the generalization of the operators $Z^{\omega}_{\alpha,\beta}, E_{\alpha,\beta}, \boxplus_{\alpha,\beta}, \boxtimes_{\alpha,\beta}$.

1. Introduction

The theory of fuzzy sets (FSs), proposed by Zadeh[20], has gained successful applications in various fields. However, the membership function of the fuzzy set is a single value between zero and one, which combines the favoring evidence and the opposing evidence. According to Fuzzy Set Theory, if the membership degree of an element x is $\mu(x)$, the nonmembership degree is $1-\mu(x)$; thus, it is fixed.

Intuitionistic fuzzy sets were introduced by Atanassov in 1983 [1] and formed an extension of fuzzy sets by enlarging the truth value set to the lattice $[0,1]\times[0,1]$ as defined below.

Definition 1.1. Let $L=[0,1]$ then $L^*=\{(x_1,x_2)\in[0,1]^2 : x_1+x_2\leq 1\}$ is a lattice with

$$(x_1,x_2)\leq(y_1,y_2) :\Longleftrightarrow \text{“}x_1\leq y_1 \text{ and } x_2\geq y_2\text{”}$$

The units of this lattice are denoted by $0_{L^*}=(0,1)$ and $1_{L^*}=(1,0)$. The lattice $(L^*,\leq)$ is a complete lattice: For each $A\subseteq L^*$,

$$supA=(sup\{x\in[0,1] : y\in[0,1],(x,y)\in A\}, inf\{y\in[0,1] : x\in[0,1],(x,y)\in A\})$$

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and

$infA = (inf\{x \in [0,1] : y \in [0,1], (x,y) \in A\}, sup\{y \in [0,1] : x \in [0,1], (x,y) \in A\})$

As it is well known, every lattice $(L^*, \leq)$ has an equivalent definition as an algebraic structure $(L, \wedge, \vee)$ where the meet operator "$\wedge$" and the join operator "$\vee$" are linked to the ordering " $\leq$ " as in the following equivalence, for $x, y \in L^*$,

$$x \leq y \Longleftrightarrow x \vee y = y \Longleftrightarrow x \wedge y = x$$

The operators $\wedge$ and $\vee$ (join and meets resp.) on $(L^*, \leq)$ are defined as follows, for $(x_1, y_1), (x_2, y_2) \in L^*$:

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \vee y_2)$$

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2)$$

Definition 1.2. [1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A ", $\nu_A(x), (\nu_A : X \to [0,1])$is called the " degree of non- membership of x in A ",and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X.$$

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 1.3. [1]An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 1.4. [1]Let $A \in IFS$ and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ then the set

$$A^c = \{< x, \nu_A(x), \mu_A(x) >: x \in X\}$$

is called the complement of A.

The modal operators have been known to be important tools for IFSs where the operators are defined on the contrary to the FSs. Intuitionistic fuzzy operators and some properties of these operators were examined by several authors [6], [7], [13], [14]. In addition, the fuzzy closer operators, the intuitionistic fuzzy topological operators defined on the mathematical structure have also been studied [15], [16], [18], [19].

The notion of Intuitionistic Fuzzy Operators was firstly introduced by Atanassov [2]. The simplest one among them is presented as in the following definition.

Definition 1.5. [3] Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X), \alpha, \beta \in [0,1]$.

(1) $\boxplus A = \{< x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2} >: x \in X\}$
(2) $\boxtimes A = \{< x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2} >: x \in X\}$

After this definition, in 2001, Atanassov, in [4], defined the following extension of these operators:

Definition 1.6. [4] Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta \in [0,1]$.

(1) $\boxplus_\alpha A = \{< x, \alpha\mu_A(x), \alpha\nu_A(x) + 1 - \alpha >: x \in X\}$
(2) $\boxtimes_\alpha A = \{< x, \alpha\mu_A(x) + 1 - \alpha, \alpha\nu_A(x) >: x \in X\}$

In these operators $\boxplus_\alpha$and $\boxtimes_\alpha$; If we choose $\alpha = \frac{1}{2}$, we get the operators $\boxplus$, $\boxtimes$, resp. Therefore, the operators $\boxplus_\alpha$and $\boxtimes_\alpha$ are the extensions of the operators $\boxplus$, $\boxtimes$, resp. Some relationships between these operators were studied by several authors ([14],[17]) In 2004, the second extension of these operators was introduced by Dencheva in [14].

Definition 1.7. [14] Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta \in [0,1]$.

(1) $\boxplus_{\alpha,\beta} A = \{< x, \alpha\mu_A(x), \alpha\nu_A(x) + \beta >: x \in X\}$ where $\alpha + \beta \in [0,1]$.
(2) $\boxtimes_{\alpha,\beta} A = \{< x, \alpha\mu_A(x) + \beta, \alpha\nu_A(x) >: x \in X\}$where $\alpha + \beta \in [0,1]$.

The concepts of the modal operators are introduced and studied by different researchers, [4, 5, 6, 7, 8], [14], [15], [17], etc.

In 2006, the third extension of the above operators was studied by Atanassov . He defined the following operators in [6]

Definition 1.8. [6]Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$.

(1) $\boxplus_{\alpha,\beta,\gamma}(A) = \{< x, \alpha\mu_A(x), \beta\nu_A(x) + \gamma >: x \in X\}$ where $\alpha, \beta, \gamma \in [0,1], \max\{\alpha, \beta\} + \gamma \leqslant 1$.
(2) $\boxtimes_{\alpha,\beta,\gamma}(A) = \{< x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) >: x \in X\}$ where $\alpha, \beta, \gamma \in [0,1], \max\{\alpha, \beta\} + \gamma \leqslant 1$.

If we choose $\alpha = \beta$ and $\gamma = \beta$ in the above operators, then we can see easily that $\boxplus_{\alpha,\alpha,\gamma} = \boxplus_{\alpha,\beta}$and $\boxtimes_{\alpha,\alpha,\gamma} = \boxtimes_{\alpha,\beta}$.Therefore, we can say that $\boxplus_{\alpha,\beta,\gamma}$and $\boxtimes_{\alpha,\beta,\gamma}$are the extensions of the operators $\boxplus_{\alpha,\beta}$, $\boxtimes_{\alpha,\beta}$,resp. From these extensions, we get the first diagram of one type modal operators over Intuitionistic Fuzzy Sets as displayed in Figure1.

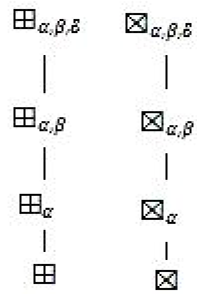

FIGURE 1

In 2007, after this diagram, the author [11] defined a new operator and studied some of its properties. This operator is named $E_{\alpha,\beta}$ and defined as follows:

Definition 1.9. [11]Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta \in [0,1]$. We define the following operator:

$$E_{\alpha,\beta}(A) = \{< x, \beta(\alpha\mu_A(x) + 1 - \alpha), \alpha(\beta\nu_A(x) + 1 - \beta) >: x \in X\}$$

If we choose $\alpha = 1$ and subtitute α for β we get the operator $\boxplus_\alpha$. Similarly, if $\beta = 1$ is chosen and substituted for β, we get the operator $\boxtimes_\alpha$. In the wiev of this definition, the diagram of one type modal operators on IFSs is figured below:

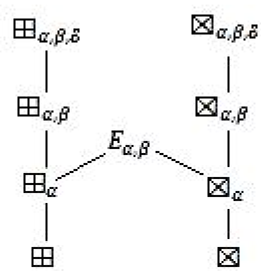

FIGURE 2

These extensions have been investigated by several authors [14], [7, 8, 9]. In particular, the authors have made significant contributions to these operators. In 2007, Atanassov introduced the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ which is a natural extension of all these operators in [7].

Definition 1.10. [7] Let Xbe a set, A$\in IFS(X)$, $\alpha, \beta, \gamma, \delta \in [0,1]$ such that

$$\max(\alpha, \beta) + \gamma + \delta \leqslant 1$$

then the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ defined by

$$\boxdot_{\alpha,\beta,\gamma,\delta}(A) = \{< x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta >: x \in X\}$$

This operator changes the one type modal operators' diagram as in fig3,

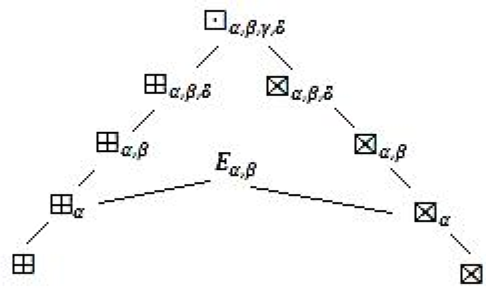

FIGURE 3

At the end of the these studies, Atanassov though that this diagram was completed. However, he realized that it wasn't totally true since there was an operator which was also an extension of two type modal oparators.

In 2008, he defined this most general operator $\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$as following:

Definition 1.11. [8] Let X be a set,$A \in IFS(X)$, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ such that

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leqslant 1$$

and

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0$$

then the operator $\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ defined by

$$\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A) = \{< x, \alpha\mu_A(x) - \varepsilon\nu_A(x) + \gamma, \beta\nu_A(x) - \zeta\mu_A(x) + \delta >: x \in X\}$$

After this definition, the one type modal operators' diagram becomes as in Figure 4.

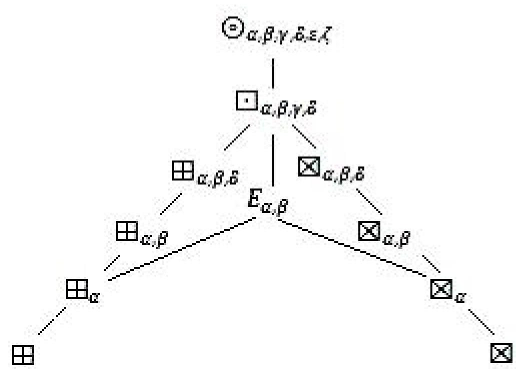

FIGURE 4

In 2010, the author [12] defined a new operator which is a generalization of $E_{\alpha,\beta}$.

Definition 1.12. [12]Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta, \omega \in [0, 1]$. We define the following operator:

$$Z^{\omega}_{\alpha,\beta}(A) = \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta) >: x \in X\}$$

Likewise the other one type modal operators, some properties of the operator $Z^{\omega}_{\alpha,\beta}$ were studied in [12] and [13].

Proposition 1.13. [12]*Let* $\alpha, \beta \in I$, $\alpha \geq \beta$ *and let* $A \in IFS(X)$ *then*

(1) $Z^{1}_{\alpha,\beta}(A) = E_{\alpha,\beta}$
(2) $Z^{1}_{\frac{1}{2},1}(A) = \boxtimes A$
(3) $Z^{1}_{1,\frac{1}{2}}(A) = \boxplus A$
(4) $Z^{1}_{\alpha,1}(A) = \boxtimes_{\alpha} A$
(5) $Z^{1}_{1,\alpha}(A) = \boxplus_{\alpha} A$

Theorem 1.14. [12]*Let* $\alpha, \beta, \omega \in I$, *and let* $A \in IFS(X)$.

(1) *If* $\alpha \geq \beta$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega}_{\beta,\alpha}(A)$
(2) *If* $\lambda, \tau \in I$, $\tau \geq \beta, \alpha \geq \lambda, \alpha\beta = \lambda\tau$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega}_{\lambda,\tau}(A)$
(3) *If* $\theta \in I, \alpha\beta\theta = \omega$ *then* $Z^{\omega}_{\alpha,\beta}(Z^{\theta}_{\beta,\alpha}(A)) = Z^{\omega}_{\beta,\alpha}(Z^{\theta}_{\alpha,\beta}(A))$
(4) *If* $\theta \in I$, $\alpha \geq \beta, \alpha - \alpha\beta \geq \omega \geq \beta - \alpha\beta, \alpha - \alpha\beta \geq \theta \geq \beta - \alpha\beta$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\theta}_{\beta,\alpha}(A)$
(5) $Z^{\omega}_{\alpha,\beta}(A) = (Z^{\omega}_{\beta,\alpha}(A^c))^c$
(6) *If* $\alpha \neq 0$ *and* $\beta \leqslant \omega$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\beta}_{\alpha,\omega}(A)$ *and* $Z^{\alpha}_{\omega,\beta}(A) \sqsubseteq Z^{\alpha}_{\beta,\omega}(A)$
(7) *If* $\beta \neq 0$ *and* $\omega \leqslant \alpha$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\alpha}_{\omega,\beta}(A)$ *and* $Z^{\beta}_{\alpha,\omega}(A) \sqsubseteq Z^{\beta}_{\omega,\alpha}(A)$
(8) *If* $\omega \neq 0$ *and* $\beta \leqslant \alpha$ *then* $Z^{\alpha}_{\omega,\beta}(A) \sqsubseteq Z^{\beta}_{\omega,\alpha}(A)$ *and* $Z^{\beta}_{\alpha,\omega}(A) \sqsubseteq Z^{\alpha}_{\beta,\omega}(A)$
(9) *If* $\beta \leqslant \alpha \leqslant \omega$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\alpha}_{\beta,\omega}(A)$ *and* $Z^{\alpha}_{\omega,\beta}(A) \sqsubseteq Z^{\beta}_{\alpha,\omega}(A)$
(10) *If* $\alpha \leqslant \beta \leqslant \omega$ *then* $Z^{\beta}_{\omega,\alpha}(A) \sqsubseteq Z^{\alpha}_{\beta,\omega}(A)$
(11) *If* $\beta \leqslant \omega \leqslant \alpha$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\beta}_{\omega,\alpha}(A)$
(12) *If* $\theta \in I, \beta \neq 1, \alpha \geq \beta, \theta > \omega, \alpha^2 \geq \frac{1-\alpha}{1-\beta}$ $A \in IFS(X)$ *then* $Z^{\omega}_{\beta,\alpha}(Z^{\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\theta}_{\beta,\alpha}(Z^{\omega}_{\alpha,\beta}(A))$
(13) *If* $\alpha \neq 1$, $0 \leqslant \alpha + \beta \leqslant 1$ *then* $Z^{\frac{\beta}{1-\alpha}}_{1,\alpha}(A) = \boxplus_{\alpha,\beta}(A)$ *and* $Z^{\frac{\beta}{1-\alpha}}_{\alpha,1}(A) = \boxtimes_{\alpha,\beta}(A)$
(14) *If* $\alpha + \beta \leqslant 1$ *then*$Z^{\omega}_{\alpha,\beta}$*represents* $\boxplus_{\alpha,\beta}$ *and* $\boxtimes_{\alpha,\beta}$
(15) *If we write* $\alpha\beta, \beta\omega(1 - \alpha), \alpha\beta, \alpha\omega(1 - \beta)$ *instead of* $\alpha, \beta, \gamma, \delta$ *resp. then* $\boxdot_{\alpha,\beta,\gamma,\delta}$ *represents* $Z^{\omega}_{\alpha,\beta}$.

The opposite of the Theorem1.14.(15) is not valid. Regarding the properties represented above, the diagram of one type modal operators over IFSs is displayed in Figure 5.

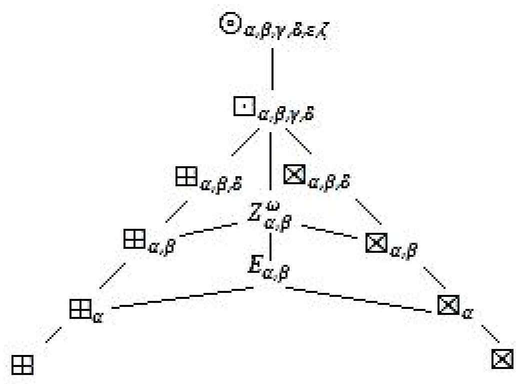

FIGURE 5

2. Some Relationships of One Type Modal Operators Over Intuitionistic Fuzzy Sets

In [13], the relationships between some One Type Modal Operators over Intuitionistic Fuzzy Sets have been studied. Some important relations are as follows:

Theorem 2.1. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2 \in I, \theta_1 + \theta_2 \in I$ *and* $A \in IFS(X)$ *then*

(1) $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A))$
(2) $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_1}(A))$, *for* $\theta_1 \leq \theta_2$

Corollary 2.2. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2 \in I, \max\{\theta_1, \theta_2\} + \theta_1 \in I, A \in IFS(X)$

(1) $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_1}(A))$
(2) *If* $\theta_1 \leq \theta_2$ *then* $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_2,\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_2,\theta_1,\theta_1}(A))$
(3) *If* $\theta_2 \leq \theta_1$ *then* $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_2,\theta_1,\theta_1}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_2,\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_2}(A))$

Theorem 2.3. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2 \in I, \theta_1 + \theta_2 \in I, A \in IFS(X)$ *then*

$$Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1}(A))$$

Theorem 2.4. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2, \theta_3 \in I, \max\{\theta_1, \theta_2\} + \theta_3 \in I$, $A \in IFS(X)$.

(1) *If* $\theta_2 \leq \min\{\theta_1, \theta_3\}$ *then* $Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1,\theta_2,\theta_3}(A))$
(2) *If* $\theta_1 \leq \theta_2$ *then* $Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1\theta_2\theta_3}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1}(A))$

Corollary 2.5. [13]*Let* $\alpha, \beta, \omega, \theta, \tau \in I, \theta \leq \tau$ *and let* $A \in IFS(X)$ *then*

$$Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta,\theta}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta,\theta,\tau}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta}(A))$$

Theorem 2.6. [13]*Let* $A, B \in IFS(X)$ *then* $A \sqsubseteq B \Rightarrow Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega}_{\alpha,\beta}(B)$

Theorem 2.7. [13]*Let* $\alpha, \beta, \omega \in I$ *and* $A, B \in IFS(X)$ *then*

(1) $Z^{\omega}_{\alpha,\beta}(A \cap B) = Z^{\omega}_{\alpha,\beta}(A) \cap Z^{\omega}_{\alpha,\beta}(B)$
(2) $Z^{\omega}_{\alpha,\beta}(A \cup B) = Z^{\omega}_{\alpha,\beta}(A) \cup Z^{\omega}_{\alpha,\beta}(B)$

Theorem 2.8. [13]*Let* $\alpha, \beta, \omega \in I, \beta \leq \alpha$ *and* $A \in IFS(X)$ *then*

$$Z^{\omega}_{\alpha,\beta}(Z^{\omega}_{\beta,\alpha}(A)) \sqsubseteq Z^{\omega}_{\beta,\alpha}(Z^{\omega}_{\alpha,\beta}(A))$$

Corollary 2.9. [13]*Let* $\alpha, \beta, \omega \in I, \alpha\beta \leq \omega, \alpha \leq \beta$ *and* $A \in IFS(X)$ *then*

$$Z^{\omega}_{\alpha,\beta}(E_{\beta,\alpha}(A)) \sqsubseteq Z^{\omega}_{\beta,\alpha}(E_{\alpha,\beta}(A))$$

Theorem 2.10. [13]*Let* $\alpha, \beta, \omega, \theta \in I,\ A \in IFS(X)$ *then*

(1) $Z^{\omega}_{\alpha,\beta}(\boxplus(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes(A))$
(2) $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta}(A))$
(3) $\theta_1 + \theta_2 \leq 1 \Rightarrow Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1,\theta_2}(A))$
(4) $\max\{\theta_1 + \theta_2\} + \theta_3 \leq 1 \Rightarrow Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_3}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1,\theta_2,\theta_3}(A))$

Corollary 2.11. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2 \in I, \theta_1 + \theta_2 \in I$ *and let* $A \in IFS(X)$ *then*

$$Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1}(A))$$

Theorem 2.12. [13]*Let* $\alpha, \beta, \omega, \theta \in I,\ A \in IFS(X)$ *then*

(1) $\boxplus(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes(Z^{\omega}_{\alpha,\beta}(A))$
(2) $\boxplus_{\theta}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta}(Z^{\omega}_{\alpha,\beta}(A))$
(3) $\theta_1 + \theta_2 \in I \Rightarrow \boxplus_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A))$
(4) $\max\{\theta_1 + \theta_2\} + \theta_3 \in I \Rightarrow \boxplus_{\theta_1,\theta_2,\theta_3}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta_1,\theta_2,\theta_3}(Z^{\omega}_{\alpha,\beta}(A))$

Theorem 2.13. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2, \theta_1 + \theta_2 \in I$ *and let* $A \in IFS(X)$ *then*

(1) $\boxplus_{\theta_1}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxplus_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A))$
(2) $\boxtimes_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta_1}(Z^{\omega}_{\alpha,\beta}(A))$

Corollary 2.14. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2 \in I,\ \theta_1 + \theta_2 \in I,\ A \in IFS(X)$.

(1) $\boxplus_{\theta_1}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxplus_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta_1}(Z^{\omega}_{\alpha,\beta}(A))$
(2) $E_{\alpha,\beta}(\boxplus_{\theta_1}(A)) \sqsubseteq E_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq E_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_1}(A))$
(3) $\alpha, \beta, \theta, \tau \in I, \theta \leq \tau$ *and let* $A \in IFS(X)$ *then*

$$E_{\alpha,\beta}(\boxtimes_{\theta,\theta}(A)) \sqsubseteq E_{\alpha,\beta}(\boxtimes_{\theta,\theta,\tau}(A)) \sqsubseteq E_{\alpha,\beta}(\boxtimes_{\theta}(A))$$

Theorem 2.15. [13]*Let* $\alpha, \beta, \omega \in I, A \in IFS(X)$ *then*

(1) $I_{\mu}(Z^{\omega}_{\alpha,\beta}(A)) = Z^{\omega}_{\alpha,\beta}(I_{\mu}(A))$
(2) $Z^{\omega}_{\alpha,\beta}(I_{\nu}(A)) \sqsubseteq I_{\nu}(Z^{\omega}_{\alpha,\beta}(A))$

3. The One Type Modal Operator $Z^{\omega\theta}_{\alpha\beta}$

It can be asked that whether or not there is any one type modal operator on IFs different from the others and is it possible to extend the last one type modal operators diagram. The answer is "yes". In this study, we have defined a new one type modal operator on IFS, that is generalization of the some one type modal operators. The new operator defined as follows:

Definition 3.1. Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta, \omega, \theta \in [0, 1]$. We define the following operator:

$$Z^{\omega,\theta}_{\alpha,\beta}(A) = \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta) >: x \in X\}$$

If A$\in IFS(X)$ then $\mu_A(x) + \nu_A(x) \leq 1$ for every $x \in X$. For $\alpha, \beta \in I$ if we use $\alpha\beta - \beta \leq 0, \alpha\theta - 1 \leq 0$ and $1 - \omega \geq 0, 1 - \beta \geq 0$ then

$$(\alpha\beta - \beta)(1 - \omega) + (\alpha\theta - 1)(1 - \beta) \leq 0.$$

Thus,

$$\begin{aligned}
&\alpha\beta(\mu_A(x) + \nu_A(x)) + \beta\omega - \alpha\beta\omega + \alpha\theta - \alpha\beta\theta \\
&\leq \alpha\beta + \beta\omega - \alpha\beta\omega + \alpha\theta - \alpha\beta\theta \\
&= \alpha\beta + \beta\omega - \alpha\beta\omega + \alpha\theta - \alpha\beta\theta + \beta - \beta + 1 - 1 \\
&= \alpha\beta(1 - \omega) - \beta(1 - \omega) + \alpha\theta(1 - \beta) - (1 - \beta) + 1 \\
&= (\alpha\beta - \beta)(1 - \omega) + (\alpha\theta - 1)(1 - \beta) + 1 \\
&\leq 1
\end{aligned}$$

therefore if $A \in IFS(X)$ then $Z^{\omega,\theta}_{\alpha,\beta}(A) \in IFS(X)$. It is clear that

$$Z^{\omega,\omega}_{\alpha,\beta}(A) = Z^{\omega}_{\alpha,\beta}(A)$$

Similar to other operators, this operator has some properties and relationships by itself. In this section, we study these properties.

Theorem 3.2. *Let $\alpha, \beta, \omega, \theta \in I$, $\omega \leq \theta$ and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega}_{\alpha,\beta}(A)$$

Proof. If we use $\omega \leq \theta$ then $\alpha\omega \leq \alpha\theta$.

$$\begin{aligned}
Z^{\omega,\theta}_{\alpha,\beta}(A) &= \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta) >: x \in X\} \\
&\sqsubseteq \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta) >: x \in X\} \\
&= Z^{\omega}_{\alpha,\beta}(A)
\end{aligned}$$

□

Theorem 3.3. *Let $\alpha, \beta, \omega, \theta \in I$, $\omega \leq \theta$ and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\theta,\omega}_{\alpha,\beta}(A)$$

Proof. If we use $\omega \leq \theta$ and $\alpha\beta \leq \beta$ then $\alpha\beta\left(\theta-\omega\right) \leq \beta\left(\theta-\omega\right).$

$$\begin{aligned} \beta\omega-\alpha\beta\omega &\leq \beta\theta-\alpha\beta\theta \\ &\Rightarrow \alpha\beta\mu_A(x)+\beta\omega-\alpha\beta\omega \leq \alpha\beta\mu_A(x)+\beta\theta-\alpha\beta\theta \end{aligned}$$

now if we use $\alpha\beta \leq \alpha$ then

$$\begin{aligned} \alpha\beta\left(\theta-\omega\right) &\leq \alpha\left(\theta-\omega\right). \\ &\Rightarrow \alpha\omega-\alpha\beta\omega \leq \alpha\theta-\alpha\beta\theta \\ &\Rightarrow \alpha\beta\nu_A(x)+\alpha\omega-\alpha\beta\omega \leq \alpha\beta\nu_A(x)+\alpha\theta-\alpha\beta\theta \end{aligned}$$

So,

$$\begin{aligned} Z^{\omega,\theta}_{\alpha,\beta}(A) &= \{<x, \beta(\alpha\mu_A(x)+\omega-\omega.\alpha), \alpha(\beta\nu_A(x)+\theta-\theta.\beta)>: x \in X\} \\ &\sqsubseteq \{<x, \beta(\alpha\mu_A(x)+\theta-\theta.\alpha), \alpha(\beta\nu_A(x)+\omega-\omega.\beta)>: x \in X\} \\ &= Z^{\theta,\omega}_{\alpha,\beta}(A) \end{aligned}$$

□

Theorem 3.4. *Let $\alpha, \beta, \omega, \theta \in I$, $\alpha \geq \beta$ and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega,\theta}_{\beta,\alpha}(A)$$

Proof. If we use $\beta \leq \alpha$ then $\beta\omega \leq \alpha\omega$ and $\alpha\theta \geq \beta\theta$.

$$\begin{aligned} Z^{\omega,\theta}_{\alpha,\beta}(A) &= \{<x, \beta\alpha\mu_A(x)+\beta\omega-\omega\beta\alpha, \alpha\beta\nu_A(x)+\theta\alpha-\theta\alpha\beta>: x \in X\} \\ &\sqsubseteq \{<x, \beta\alpha\mu_A(x)+\alpha\omega-\beta\alpha\omega, \alpha\beta\nu_A(x)+\theta\beta-\theta\alpha\beta>: x \in X\} \\ &= Z^{\omega,\theta}_{\beta,\alpha}(A) \end{aligned}$$

Hence,

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega,\theta}_{\beta,\alpha}(A)$$

□

Theorem 3.5. *Let $\alpha, \beta, \omega, \theta \in I$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A^c) = Z^{\theta,\omega}_{\beta,\alpha}(A)^c$$

Proof. If we use definition of A^c it is clear that

$$\begin{aligned} Z^{\omega,\theta}_{\alpha,\beta}(A^c) &= \{<x, \beta\alpha\nu_A(x)+\beta\omega-\omega\beta\alpha, \alpha\beta\mu_A(x)+\theta\alpha-\theta\alpha\beta>: x \in X\} \\ &= Z^{\theta,\omega}_{\beta,\alpha}(A)^c \end{aligned}$$

□

Theorem 3.6. *Let $\alpha, \beta, \omega, \theta \in I$, $\omega = \theta, \omega.\theta = \alpha.\beta$ and let $A \in IFS(X)$ then*

$$Z^{\beta,\alpha}_{\omega,\theta}(A) = Z^{\omega,\theta}_{\alpha,\beta}(A)$$

Proof. The proof is clear. □

Theorem 3.7. *Let $\alpha, \beta, \omega, \theta, \sigma \in I$, $\omega \leq \sigma$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\sigma,\theta}_{\alpha,\beta}(A)$$

Proof. If we use $\omega \leq \sigma$ then $\omega\beta(1-\alpha) \leq \sigma\beta(1-\alpha)$. With this inequality

$$\begin{aligned} Z^{\omega,\theta}_{\alpha,\beta}(A) &= \{< x, \beta\alpha\mu_A(x) + \beta\omega - \omega\beta\alpha, \alpha\beta\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\ &\sqsubseteq \{< x, \beta\alpha\mu_A(x) + \beta\sigma - \sigma\beta\alpha, \alpha\beta\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\ &= Z^{\sigma,\theta}_{\alpha,\beta}(A) \end{aligned}$$

□

Theorem 3.8. *Let $\alpha, \beta, \omega, \theta, \sigma \in I$, $\max\{\omega, \sigma\} \leq \theta$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\theta,\sigma}_{\alpha,\beta}(A)$$

Proof. If we use $\omega \leq \theta$ then

$$\begin{aligned} \omega - \omega\alpha &\leq \theta - \alpha\theta \\ &\Rightarrow \beta\alpha\mu_A(x) + \beta\omega - \alpha\beta\omega \leq \beta\alpha\mu_A(x) + \beta\theta - \alpha\beta\theta \end{aligned}$$

and if we use $\sigma \leq \theta$ then

$$\begin{aligned} \theta - \beta\theta &\geq \sigma - \beta\sigma \\ &\Rightarrow \alpha\beta\nu_A(x) + \theta\alpha - \theta\alpha\beta \geq \alpha\beta\nu_A(x) + \alpha\sigma - \alpha\beta\sigma \end{aligned}$$

Thus,

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\theta,\sigma}_{\alpha,\beta}(A)$$

□

Corollary 3.9. *Let $\alpha, \beta, \omega, \theta \in I$, $\omega \leq \theta$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\theta}_{\alpha,\beta}(A)$$

Theorem 3.10. *Let $\alpha, \beta \in I$, $\alpha \leq \beta$ and $A \in IFS(X)$ then*

$$Z^{\alpha,\beta}_{\alpha,\beta}(A) \sqsubseteq Z^{\beta,\alpha}_{\alpha,\beta}(A)$$

Proof. If we use $\alpha \leq \beta$ then $\alpha\beta \leq \beta^2$ and $\alpha\beta \geq \alpha^2$. So,

$$\begin{aligned} Z^{\alpha,\beta}_{\alpha,\beta}(A) &= \{< x, \beta\alpha\mu_A(x) + \beta\alpha - \beta\alpha^2, \alpha\beta\nu_A(x) + \alpha\beta - \alpha\beta^2 >: x \in X\} \\ &\sqsubseteq \{< x, \beta\alpha\mu_A(x) + \beta^2 - \beta^2\alpha, \alpha\beta\nu_A(x) + \alpha^2 - \alpha^2\beta >: x \in X\} \\ &= Z^{\beta,\alpha}_{\alpha,\beta}(A) \end{aligned}$$

□

Theorem 3.11. *Let $\alpha, \beta, \lambda, \tau, \theta, \omega \in I$, $\tau \geq \beta, \alpha \geq \lambda, \omega \leq \sigma, \theta \geq \varepsilon, \alpha\beta = \lambda\tau$, and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\sigma,\varepsilon}_{\lambda,\tau}(A)$$

Proof. If we use the above inequalities

$$\begin{aligned} \beta\omega(1-\alpha) &\leq \tau\sigma(1-\lambda) \\ &\Rightarrow \beta\omega \leq \alpha\beta\omega + \tau\sigma - \tau\sigma\lambda \\ &\Rightarrow \beta\alpha\mu_A(x) + \beta\omega - \alpha\beta\omega \leq \lambda\tau\mu_A(x) + \tau\sigma - \tau\sigma\lambda \end{aligned}$$

and

$$\begin{aligned} \alpha\theta\,(1-\beta) &\geq \lambda\varepsilon\,(1-\tau) \\ &\Rightarrow \alpha\beta\nu_A(x)+\theta\alpha-\theta\alpha\beta \geq \lambda\tau\nu_A(x)+\lambda\varepsilon-\tau\lambda\varepsilon \end{aligned}$$

Therefore,

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\sigma,\varepsilon}_{\lambda,\tau}(A)$$

□

Corollary 3.12. *Let $\alpha,\beta,\lambda,\tau,\theta,\omega \in I$, $\tau \geq \beta, \alpha \geq \lambda, \alpha\beta = \lambda\tau$ and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega,\theta}_{\lambda,\tau}(A)$$

Theorem 3.13. *Let $\alpha,\beta,\omega,\theta \in I$ and $A,B \in IFS(X)$ then*

(1) $Z^{\omega,\theta}_{\alpha,\beta}(A\cap B) = Z^{\omega,\theta}_{\alpha,\beta}(A) \cap Z^{\omega,\theta}_{\alpha,\beta}(B)$
(2) $Z^{\omega,\theta}_{\alpha,\beta}(A\cup B) = Z^{\omega,\theta}_{\alpha,\beta}(A) \cup Z^{\omega,\theta}_{\alpha,\beta}(B)$

Proof. We know that for every $x,y,c \in I$ $\min\{x+c,y+c\} = \min\{x,y\}+c$ and $\max\{x+c,y+c\} = \max\{x,y\}+c$. (1)If we use the above equalities then we get,

$$\begin{gathered} Z^{\omega,\theta}_{\alpha,\beta}(A) \cap Z^{\omega,\theta}_{\alpha,\beta}(B) \\ = \{<x, \min(\alpha\beta\mu_A(x)+\beta\omega-\alpha\beta\omega, \alpha\beta\mu_B(x)+\beta\omega-\alpha\beta\omega), \\ \max\{\alpha\beta\nu_A(x)+\alpha\theta-\alpha\beta\theta, \alpha\beta\nu_B(x)+\alpha\theta-\alpha\beta\theta\} >: x \in X\} \\ = \{<x, \min(\alpha\beta\mu_A(x), \alpha\beta\mu_B(x))+\beta\omega-\alpha\beta\omega, \\ \max(\alpha\beta\nu_A(x), \alpha\beta\nu_B(x))+\alpha\theta-\alpha\beta\theta >: x \in X\} \\ = Z^{\omega}_{\alpha,\beta}(A\cap B) \\ Z^{\omega,\theta}_{\alpha,\beta}(A\cap B) = Z^{\omega,\theta}_{\alpha,\beta}(A) \cap Z^{\omega,\theta}_{\alpha,\beta}(B) \end{gathered}$$

(2) If we use the same equalities as above then we get,

$$\begin{gathered} Z^{\omega,\theta}_{\alpha,\beta}(A) \cup Z^{\omega,\theta}_{\alpha,\beta}(B) \\ = \{<x, \max(\alpha\beta\mu_A(x)+\beta\omega-\alpha\beta\omega, \alpha\beta\mu_B(x)+\beta\omega-\alpha\beta\omega), \\ \min\{\alpha\beta\nu_A(x)+\alpha\theta-\alpha\beta\theta, \alpha\beta\nu_B(x)+\alpha\theta-\alpha\beta\theta\} >: x \in X\} \\ = \{<x, \max(\alpha\beta\mu_A(x), \alpha\beta\mu_B(x))+\beta\omega-\alpha\beta\omega, \\ \min(\alpha\beta\nu_A(x), \alpha\beta\nu_B(x))+\alpha\theta-\alpha\beta\theta >: x \in X\} \\ = Z^{\omega,\theta}_{\alpha,\beta}(A\cup B) \end{gathered}$$

So, we get

$$Z^{\omega}_{\alpha,\beta}(A\cup B) = Z^{\omega}_{\alpha,\beta}(A) \cup Z^{\omega}_{\alpha,\beta}(B)$$

□

Theorem 3.14. *Let $\alpha,\beta,\omega,\theta \in I$, $\beta \leq \alpha$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

Proof. If we use $\beta \leq \alpha$ then $\beta^2 \leq \alpha\beta$ and $\alpha^2 \geq \alpha\beta$ So

$$\begin{aligned}\alpha\beta^2 + \beta^2 &\leq \alpha\beta + \alpha^2\beta \\ &\Rightarrow \alpha\beta^2 - \alpha\beta \leq \alpha^2\beta - \beta^2 \\ &\Rightarrow \alpha^2\beta^2\mu_A(x) + \alpha\beta^2\omega - \alpha^2\beta^2\omega + \beta\omega - \alpha\beta\omega\end{aligned}$$

$$\leq \alpha^2\beta^2\mu_A(x) + \alpha^2\beta\omega - \alpha^2\beta^2\omega + \beta\omega - \beta^2\omega$$

on the other hand

$$\begin{aligned}\alpha^2\beta + \alpha^2 &\geq \alpha\beta^2 + \alpha\beta \\ &\Rightarrow \alpha^2\beta\theta - \alpha\beta\theta \geq \alpha\beta^2\theta - \alpha^2\theta \\ &\Rightarrow \alpha^2\beta^2\nu_A(x) + \alpha^2\beta\theta - \alpha^2\beta^2\theta + \alpha\theta - \alpha\beta\theta\end{aligned}$$

$$\geq \alpha^2\beta^2\nu_A(x) + \alpha\beta^2\theta - \alpha^2\beta^2\theta + \alpha\theta - \alpha^2\theta$$

Hence,

$$Z^{\omega,\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

□

Theorem 3.15. *Let* $\alpha, \beta, \omega, \theta \in I$, $\omega \geq \theta$, $\theta(1-\theta) \geq \omega(1-\beta)$ *and* $A \in IFS(X)$ *then*

$$Z^{\omega,\theta}_{\alpha,\beta}(Z^{\alpha,\beta}_{\omega,\theta}(A)) \sqsubseteq Z^{\alpha,\beta}_{\omega,\theta}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Proof. The proof is clear. □

Theorem 3.16. *Let* $\alpha, \beta, \omega, \theta \in I$, $\beta \leq \alpha, \alpha.\beta.\omega \leq \theta$ *and* $A \in IFS(X)$ *then*

$$Z^{\theta,\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A)) \sqsubseteq Z^{\theta,\omega}_{\beta,\alpha}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Proof. If we use $\beta \leq \alpha$ and $\alpha.\beta.\omega \leq \theta$ we get

$$\alpha\beta\omega(\alpha - \beta) \leq \theta(\alpha - \beta)$$

with this inequality

$$Z^{\theta,\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

$$= \{< x, \beta^2\alpha^2\mu_A(x) + \alpha^2\beta\omega - \beta^2\alpha^2\omega + \beta\theta - \alpha\beta\theta,$$

$$\beta^2\alpha^2\nu_A(x) + \alpha\beta^2\theta - \alpha^2\beta^2\theta + \alpha\omega - \alpha\beta\omega >: x \in X\}$$

$$\sqsubseteq \{< x, \beta^2\alpha^2\mu_A(x) + \alpha\beta^2\omega - \beta^2\alpha^2\omega + \alpha\theta - \alpha\beta\theta,$$

$$\beta^2\alpha^2\nu_A(x) + \alpha^2\beta\theta - \alpha^2\beta^2\theta + \beta\omega - \alpha\beta\omega >: x \in X\}$$

$$= Z^{\theta,\omega}_{\beta,\alpha}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

□

In wiev of this properties, we can say that the operator $Z^{\omega,\theta}_{\alpha,\beta}$ is a generalization of $Z^{\omega}_{\alpha,\beta}$,and also, $E_{\alpha,\beta}, \boxplus_{\alpha,\beta}, \boxtimes_{\alpha,\beta}$. If we use this result, we get the new diagram of one type modal operators as in Figure 6:

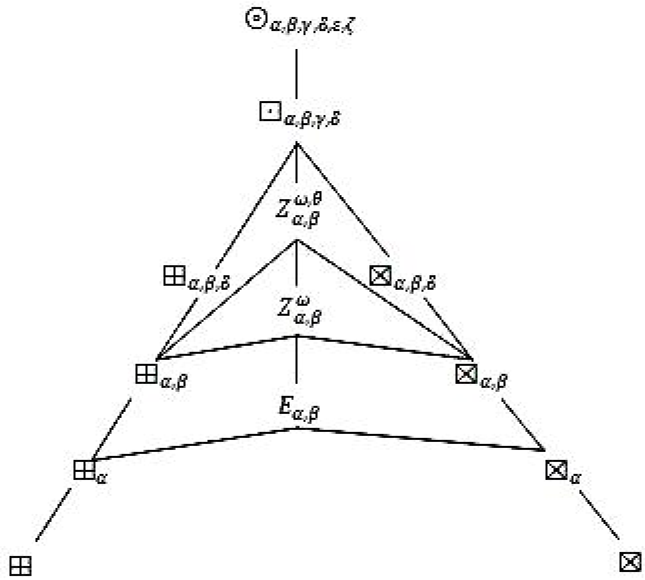

FIGURE 6

This is the last diagram of one type modal operators on IFSs. If we use the above properties, we can say that there is no one type modal operator on IFSs so that it is written inside of the last diagram. On the other hand, in the future, we may say that there are some one type modal operators that can be written outside of the last diagram.

4. Some Relationships Among One Type Modal Operators on Intuitionistic Fuzzy Sets

In several papers, some authors have discussed the relationships among the one type modal operators, some of which are given in the above section. First and foremost, we want to study standard relationships between the operator $Z^{\omega,\theta}_{\alpha,\beta}$ and the others. Afterwards, we will show the different relationships between them.

Theorem 4.1. *Let $\alpha, \beta, \omega, \theta \in I, \alpha \geq \beta$ and $A \in IFS(X)$ then*

$$Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

Proof. If we use $\beta \leq \alpha$ then $\alpha\beta^2 \leq \alpha^2\beta$

$$\begin{aligned} &Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \\ &= \{< x, \beta^2\alpha^2\mu_A(x) + \alpha\beta^2\omega - \beta^2\alpha^2\omega + \beta\omega - \alpha\beta\omega, \\ &\beta^2\alpha^2\nu_A(x) + \alpha^2\beta\theta - \alpha^2\beta^2\theta + \alpha\omega - \alpha\beta\omega >: x \in X\} \\ &\sqsubseteq \{< x, \beta^2\alpha^2\mu_A(x) + \alpha^2\beta\omega - \beta^2\alpha^2\omega + \beta\omega - \alpha\beta\omega, \\ &\beta^2\alpha^2\nu_A(x) + \alpha\beta^2\theta - \alpha^2\beta^2\theta + \alpha\omega - \alpha\beta\omega >: x \in X\} \end{aligned}$$

$$= Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

Thus,

$$Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

□

Theorem 4.2. *Let $\alpha, \beta, \omega, \theta \in I, \alpha \geq \beta$ and $A \in IFS(X)$ then*

$$Z^{\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\theta}_{\beta,\alpha}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Proof. If we use $\alpha \geq \beta$ then $\alpha\theta \geq \beta\theta$

$$Z^{\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

$$= \{< x, \beta^2\alpha^2\mu_A(x) + \alpha\beta^2\omega - \beta^2\alpha^2\omega + \beta\theta - \alpha\beta\theta,$$

$$\beta^2\alpha^2\nu_A(x) + \alpha^2\beta\theta - \alpha^2\beta^2\theta + \alpha\theta - \alpha\beta\theta >: x \in X\}$$

$$\sqsubseteq \{< x, \beta^2\alpha^2\mu_A(x) + \alpha\beta^2\omega - \beta^2\alpha^2\omega + \alpha\theta - \alpha\beta\theta,$$

$$\beta^2\alpha^2\nu_A(x) + \alpha^2\beta\theta - \alpha^2\beta^2\theta + \beta\theta - \alpha\beta\theta >: x \in X\}$$

$$= Z^{\theta}_{\beta,\alpha}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Therefore,

$$Z^{\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\theta}_{\beta,\alpha}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

□

Theorem 4.3. *Let $\alpha, \beta, \omega, \theta, \tau \in I$, $A \in IFS(X)$ then*

(1) $Z^{\omega,\theta}_{\alpha,\beta}(\boxplus(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes(A))$
(2) $Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau}(A))$
(3) $\tau_1 + \tau_2 \leq 1 \Rightarrow Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau_1,\tau_2}(A))$
(4) $\max\{\tau_1, \tau_2\} + \tau_3 \leq 1 \Rightarrow Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2,\tau_3}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau_1,\tau_2,\tau_3}(A))$

Proof. (1)If we use definitions of $\boxplus(A)$ and $\boxtimes(A)$ then

$$\begin{aligned} Z^{\omega,\theta}_{\alpha,\beta}(\boxplus(A)) &= \{< x, \beta\alpha\frac{\mu_A(x)}{2} + \beta\omega - \omega\beta\alpha, \alpha\beta\frac{\nu_A(x)+1}{2} + \theta\alpha - \theta\alpha\beta >: x \in X\} \\ &\sqsubseteq \{< x, \beta\alpha\frac{\mu_A(x)+1}{2} + \beta\omega - \omega\beta\alpha, \alpha\beta\frac{\nu_A(x)}{2} + \theta\alpha - \theta\alpha\beta >: x \in X\} \\ &= Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes(A)) \end{aligned}$$

(2)If we use $\alpha\beta - \alpha\beta\tau \geq 0$ then

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau}(A))$$

$$= \{< x, (\beta\alpha\tau)\,\mu_A(x) + \beta\omega - \omega\beta\alpha,$$

$$(\beta\alpha\tau)\,\nu_A(x) + \alpha\beta - \alpha\beta\tau + \theta\alpha - \theta\alpha\beta >: x \in X\}$$

$$\sqsubseteq \{< x, (\beta\alpha\tau)\,\mu_A(x) + \alpha\beta - \alpha\beta\tau + \beta\omega - \omega\beta\alpha,$$

$$(\beta\alpha\tau)\,\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\}$$

$$= Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau}(A))$$

(3-4) are shown by the same way as above. □

Corollary 4.4. *Let $\alpha, \beta, \omega, \theta \in I, \beta \leq \alpha$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(E_{\alpha,\beta}(A)) \sqsubseteq Z^{\omega,\theta}_{\beta,\alpha}(E_{\alpha,\beta}(A))$$

Theorem 4.5. *Let $\alpha, \beta, \omega, \theta, \tau_1, \tau_2 \in I, \max\{\tau_1, \tau_2\} + \tau_1 \in I$ and $\tau_1 \leq \tau_2, A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2,\tau_1}(A))$$

Proof. If we use $\tau_1 + \tau_2 \leq 1$ then

$$\begin{aligned}
\alpha\beta - \alpha\beta\tau_1 &\geq \alpha\beta\tau_2 \\
&\Rightarrow (\beta\alpha\tau_1)\,\nu_A(x) + \alpha\beta - \alpha\beta\tau_1 + \theta\alpha - \theta\alpha\beta \\
&\geq (\beta\alpha\tau_1)\,\nu_A(x) + \alpha\beta\tau_2 + \theta\alpha - \theta\alpha\beta
\end{aligned}$$

so we get

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A))$$

on the other hand if we use $\tau_1 \leq \tau_2$ then

$$\begin{aligned}
\tau_2 - \tau_1 &\geq 0 \\
&\Rightarrow \tau_1\nu_A(x) + \tau_2 \geq \tau_2\nu_A(x) + \tau_1 \\
&\Rightarrow (\beta\alpha\tau_1)\,\nu_A(x) + \alpha\beta\tau_2 + \theta\alpha - \theta\alpha\beta \\
&\geq (\beta\alpha\tau_2)\,\nu_A(x) + \alpha\beta\tau_1 + \theta\alpha - \theta\alpha\beta
\end{aligned}$$

and we get

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2,\tau_1}(A))$$

So,

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2,\tau_1}(A))$$

□

Theorem 4.6. *Let $\alpha, \beta, \omega, \theta, \tau, \gamma \in I, \tau \leq \gamma$ and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau,\gamma}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau}(A))$$

Proof. If we use $\tau \leq \gamma$ then

$$\begin{aligned}
& Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau}(A)) \\
= & \{< x, (\beta\alpha\tau)\,\mu_A(x) + \alpha\beta\tau + \beta\omega - \omega\beta\alpha, (\beta\alpha\tau)\,\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\
\sqsubseteq & \{< x, (\beta\alpha\tau)\,\mu_A(x) + \alpha\beta\gamma + \beta\omega - \omega\beta\alpha, (\beta\alpha\tau)\,\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\
= & Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau,\gamma}(A))
\end{aligned}$$

on the other hand if we use $\tau + \gamma \leq 1$ then $\alpha\beta\gamma \leq \alpha\beta - \alpha\beta\tau$. with inequality, we get

$$\begin{aligned}
& Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau,\gamma}(A)) \\
= & \{< x, (\beta\alpha\tau)\,\mu_A(x) + \alpha\beta\gamma + \beta\omega - \omega\beta\alpha, (\beta\alpha\tau)\,\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\
\sqsubseteq & \{< x, (\beta\alpha\tau)\,\mu_A(x) + \alpha\beta - \alpha\beta\tau + \beta\omega - \omega\beta\alpha, \\
& (\beta\alpha\tau)\,\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\
& = Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau}(A))
\end{aligned}$$

Thus,

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau,\gamma}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau}(A))$$

□

Corollary 4.7. *Let $\alpha,\beta,\omega,\theta,\tau_1,\tau_2 \in I, \tau_1+\tau_2 \in I$ and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau_1,\tau_2}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau_1}(A))$$

Theorem 4.8. *Let $\alpha,\beta,\omega,\theta,\tau_1,\tau_2 \in I$ and let $A \in IFS(X)$ then*

$$\boxplus_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxplus_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Proof. If we use $\tau_1+\tau_2 \leq 1$ then

$$\alpha\tau_1(\beta\nu_A(x)+\theta-\beta\theta)+\tau_2 \leq \alpha\tau_1(\beta\nu_A(x)+\theta-\beta\theta)+1-\tau_1$$

and if we use inequality $\beta\tau_1(\alpha\mu_A(x)+\omega-\alpha\omega)=\beta\tau_1(\alpha\mu_A(x)+\omega-\alpha\omega)$ then we get

$$\boxplus_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxplus_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

□

Theorem 4.9. *Let $\alpha,\beta,\omega,\tau_1,\tau_2 \in I,\ \tau_1+\tau_2 \leq 1,\ A \in IFS(X)$ then*

$$\boxtimes_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Proof. If we use definition of τ_1,τ_2 we get $\tau_2 \leq 1-\tau_1$ then

$$\begin{aligned} & \boxtimes_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \\ = \ & \{<x,\beta\tau_1(\alpha\mu_A(x)+\omega-\alpha\omega)+\tau_2,\alpha\tau_1(\beta\nu_A(x)+\theta-\beta\theta)>: x \in X\} \\ \sqsubseteq \ & \{<x,\beta\tau_1(\alpha\mu_A(x)+\omega-\alpha\omega)+1-\tau_1,\alpha\tau_1(\beta\nu_A(x)+\theta-\beta\theta>: x \in X\} \\ = \ & \boxtimes_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \end{aligned}$$

Hence,

$$\boxtimes_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

□

Theorem 4.10. *Let $\alpha,\beta,\omega,\theta,\tau \in I,\ A \in IFS(X)$ then*

(1) $\boxplus(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes(Z^{\omega,\theta}_{\alpha,\beta}(A))$

(2) $\boxplus_{\tau}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau}(Z^{\omega,\theta}_{\alpha,\beta}(A))$

(3) $\tau_1+\tau_2 \in I \Rightarrow \boxplus_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A))$

(4) $\max\{\tau_1,\tau_2\}+\tau_3 \in I \Rightarrow \boxplus_{\tau_1,\tau_2,\tau_3}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1,\tau_2,\tau_3}(Z^{\omega,\theta}_{\alpha,\beta}(A))$

Proof. We know that

$$\text{if } A \in IFS(X) \text{ then } \boxplus(A) \sqsubseteq \boxtimes(A).$$

So,

$$\boxplus(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes(Z^{\omega,\theta}_{\alpha,\beta}(A)).$$

Consequently, the other properties can be shown similarly. □

Corollary 4.11. *Let $\alpha,\beta,\omega,\ \tau_1,\tau_2 \in I,\ \tau_1+\tau_2 \in I,\ A \in IFS(X)$ then*

$$\boxplus_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxplus_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

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FUZZY INTEGRO-DIFFERENTIAL EQUATIONS: DISCRETE SOLUTION AND ERROR ESTIMATION

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Abstract. This paper investigates existence and uniqueness results for the first order fuzzy integro-differential equations. Then numerical results and error bound based on the left rectangular quadrature rule, trapezoidal rule and a hybrid of them are obtained. Finally an example is given to illustrate the performance of the methods.

1. Introduction

The topics of fuzzy differential equations (FDE) and fuzzy integral equations (FIE) in both theoretical and numerical points of view have been developed in recent years. Prior to discussing fuzzy integro-differential equations and their numerical treatments, it is necessary to present a brief introduction of the previous works about FDE and FIE. Goetschel, Voxman and Kaleva (see e.g [13, 15, 16]) studied initial value problems of fuzzy differential equations for the first time. The corresponding existence results for initial and boundary value problems have been obtained for fuzzy problems in [29, 18, 17, 21, 22, 24]. The main tool in studying existence and uniqueness of the solution for fuzzy integral equations is the Banach fixed point theorem which can be found in [23, 27, 5, 7, 6]. To solve fuzzy differential and integral equations numerically, several methods have been constructed in [3, 26, 19, 11, 1, 4, 11, 12, 2, 20]. Numerical procedures based on the trapezoidal quadrature rule along with convergence results have been investigated to solve fuzzy integral equations iteratively with arbitrary kernels in [11, 12]. In [9], the authors have introduced a general form for the quadrature rules and used it to solve linear fuzzy Fredholm integral equations by iterative method. Also they have investigated error estimation of the method. Bica (see e.g [10]) applied the general form to solve nonlinear fuzzy Fredholm integral equations numerically and also the error estimation of the method and a stopping criterion of the corresponding algorithm are given.

After a preliminary section, by using lemma 6.1 in [15], a fuzzy integro-differential equation is transformed to the corresponding Volterra integral equation of the second kind in section 3. Then, by using the metric appeared in [14], an existence theorem is proved irrespective to Lipschitz constant and also integration intervals in subsection 3.1. The fuzzy Volterra integral equation is solved by three methods

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in subsection 3.2. The first method which is an explicit one uses the left rectangular quadrature rule and thus no equation is needed to be solved. The trapezoidal rule is used in the second method which is going to be an implicit method and thus a linear equation should be solved. The two rules mentioned above are combined to imply the third method which is an explicit one having an acceptable error bound. Finally error bounds of the methods are investigated and a numerical example is presented to illustrate the performance of the methods.

2. Preliminaries

Definition 2.1. (see e.g. [8]) Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of the real axis $u : \mathbb{R} \to I = [0, 1]$, satisfying the following properties:

(i) u is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$,
(ii) u is a convex fuzzy set (i.e. $u(tx+(1-t)y) \geq min\{u(x), u(y)\}, \forall t \in [0, 1], x, y \in \mathbb{R}$),
(iii) u is upper semicontinuous on $\mathbb{R}$,
(iv) $\overline{\{x \in \mathbb{R}; u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Obviously $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. Here $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$ is understood as $\mathbb{R} = \{\chi_x;\ x\ is\ a\ usual\ real\ number\}$. For $0 < r \leq 1$, r-cut of fuzzy number u is defined as $[u]^r = \{x \in \mathbb{R}; u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathbb{R}; u(x) > 0\}}$. Then it is easily established that u is a fuzzy number if and only if $[u]^r$ is a closed and bounded interval for each $r \in [0, 1]$, and $[u]^1 \neq \emptyset$ (see e.g. [13]). For $u, v \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda.u$ are defined by $[u + v]^r = [u]^r + [v]^r$ and $[\lambda.u]^r = \lambda[u]^r$, $\forall r \in [0, 1]$.
Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \bigcup\{0\}$, $D(u, v) = sup_{r\in[0,1]} max\{|u^r_- - v^r_-|, |u^r_+ - v^r_+|\}$, be the Hausdorff distance between fuzzy numbers, where $[u]^r = [u^r_-, u^r_+]$ and $[v]^r = [v^r_-, v^r_+]$. We define $\|.\| = D(., \tilde{0})$, where $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$, $\tilde{0} = \chi_{\{0\}}$. Then the following properties are satisfied (see e.g. [28])

(i) $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space,
(ii) $D(u + \nu, u + \omega) = D(\nu, \omega)$,
(iii) $D(k.u, k.\nu) = |k|D(u, \nu)$,
(ii) $D(u + \nu, \omega + e) \leq D(u, \omega) + D(\nu, e)$.

Definition 2.2. (see e.g. [9]) The function $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is called a Lipschitz function if there exists a real constant $L \geq 0$ such that, for all $x, y \in [a, b]$

$$D(f(x), f(y)) \leq L|x - y|.$$

We refer to L as the Lipschitz constant of the function f.

Definition 2.3. (see e.g [28]) Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$, $\delta : [a, b] \to \mathbb{R}_+$ and $a = x_0 < x_1 < ... < x_n = b$ be a partition of the interval $[a, b]$ with the intermediate points $\psi_i \in [x_{i-1}, x_i]$. The partition $P = \{([x_{i-1}, x_i]; \psi_i); i = 1, ..., n\}$ denoted by $P = (\Delta_n, \psi)$ is called δ-fine iff $[x_{i-1}, x_i] \subseteq (\psi_i - \delta(\psi_i), \psi_i + \delta(\psi_i))$.

Definition 2.4. (see e.g [28]) The function f is called Henstock integrable if for every $\epsilon > 0$ there exists a function $\delta : [a,b] \to \mathbb{R}_+$ such that for any δ-fine partition P, we have $D\left(\sum_{i=1}^{n}(x_i - x_{i-1}).f(\psi_i), A\right) \leq \epsilon$ for some $A \in \mathbb{R}_{\mathcal{F}}$. Then A is called the Henstock integral of f and it is denoted by $(FH)\int_a^b f(t)dt$.

The integrals used in this paper are fuzzy Riemann integral which is a particular case of the Henstock integral.

Lemma 2.5. (see e.g. [9]) **(i)** *Let f and g be Henstock integrable functions and $D(f(t), g(t))$ be Lebesgue integrable. Then*

$$D\left((FH)\int_a^b f(t)dt, (FH)\int_a^b g(t)dt\right) \leq L\int_a^b D(f(t), g(t))dt.$$

(ii) *Let the function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a Henstock integrable and bounded function. Then for every fixed point $u \in [a,b]$, the function $\phi_u : [a,b] \to \mathbb{R}_+$ defined by $\phi_u(t) = D(f(u), f(t))$ is Lebesgue integrable on $[a,b]$.*

We recall the following preliminary results from [9, 25, 10].

Theorem 2.6. (see e.g. [9]) *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a Lipschitz function with the Lipschitz constant L. Then for any partition $a = x_0 < x_1 < ... < x_n = b$ and $\xi_i \in [x_{i-1}, x_i]$, we have*

$$\begin{aligned} D\left((FH)\int_a^b f(t)dt, \sum_{i=1}^{n}(x_i - x_{i-1}).f(\xi_i)\right) &\leq \frac{L}{2}\sum_{i=1}^{n}\left((x_i - \xi_i)^2 + (\xi_i - x_{i-1})^2\right) \\ &\leq \frac{L}{2}\sum_{i=1}^{n}(x_i - x_{i-1})^2. \end{aligned}$$

Corollary 2.7. (see e.g. [9]) *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a Lipschitz function with the Lipschitz constant L. Then*

$$D\left((FH)\int_a^b f(t)dt, ((x-a).f(t) + (b-x).f(s))\right) \leq L\left[\frac{(b-a)^2}{4} + (x - \frac{a+b}{2})^2\right], \quad (1)$$

for any $x \in [a,b]$, $t \in [a,x]$ and $s \in [x,b]$.

The following corollary gives a new error bound for the fuzzy variant of the classical trapezoidal rule.

Corollary 2.8. **(i)** (see e.g. [10]) *If we put $t = a$, $s = b$ and $x = \frac{a+b}{2}$, then we obtain the error bound*

$$D\left((FH)\int_a^b f(t)dt, (\frac{b-a}{2}.f(a) + \frac{b-a}{2}.f(b))\right) \leq L\frac{(b-a)^2}{4},$$

for the fuzzy trapezoidal rule.

(ii) *(see e.g. [10].) The generalization inequality for the Lipschitz fuzzy function is*

$$D\left((FH)\int_a^b f(t)dt, \sum_{i=0}^{n-1}\frac{x_{i+1} - x_i}{2}.(f(x_i) + f(x_{i+1}))\right) \leq \frac{L(b-a)^2}{4n}. \quad (2)$$

Definition 2.9. (see e.g. [25]) Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x - y$.

Definition 2.10. (see e.g. [25]) A function $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ is called H-differentiable at $x_0 \in (a, b)$ if for $h \geq 0$ sufficiently small, there exist the H-differences $f(x_0 + h) - f(x_0)$, $f(x_0) - f(x_0 - h)$ and an element $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that

$$\lim_{h \searrow 0} D\left(\frac{f(x_0+h) - f(x_0)}{h}, f'(x_0)\right) = \lim_{h \searrow 0} D\left(\frac{f(x_0) - f(x_0-h)}{h}, f'(x_0)\right) = 0.$$

(Here h at denominator means the multiplication with $\frac{1}{h}$.)

3. Main Results

Now we consider a linear fuzzy Volterra integro-differential equation of the form

$$y'(x) = f(x) + \int_a^x k(x,t).y(t)dt, \qquad x \in I \tag{3}$$

with the initial condition

$$y(a) = y_0, \tag{4}$$

where $I = [a, b]$. Let $k : G \to \mathbb{R}$ be continuous with no sign changes being Lipschitz with respect to the second variable and $f : I \to \mathbb{R}_{\mathcal{F}}$ be continuous, where

$$G := \{(x, y) | x \in I, y \in [a, x]\} \subset I \times I.$$

A function $y : I \to \mathbb{R}_{\mathcal{F}}$ is a solution of the initial value problem (3)-(4) if and only if it is continuous and satisfies the integral equation

$$y(x) = y_0 + \int_a^x f(t)dt + \int_a^x \int_a^t k(t,s).y(s)dsdt,$$

for all $x \in I$ (see e.g. [15]).
By changing the order of the integration, we have

$$y(x) = y_0 + \int_a^x f(t)dt + \int_a^x \int_s^x k(t,s).y(s)dtds.$$

Since the function k is with no sign changes by assumption, we have

$$\int_s^x k(t,s).y(s)dt = \left(\int_s^x k(t,s)dt\right).y(s)$$

and thus

$$y(x) = g(x) + \int_a^x k_1(x,s).y(s)ds,$$

where $k_1(x,s) = \int_s^x k(t,s)dt$ and $g(x) = y_0 + \int_a^x f(s)ds$.

Lemma 3.1. *Let $k(t,s)$ be continuous in (t,s) and Lipschitz with respect to s. Then $k_1(x,s)$ is Lipschitz with respect to s.*

Proof. Let $a \le s_1 \le s_2 \le x$. Then

$$\begin{aligned}
|k_1(x,s_1)-k_1(x,s_2)| &= |\int_{s_1}^{x} k(t,s_1)dt - \int_{s_2}^{x} k(t,s_2)dt| \\
&= |\int_{s_1}^{s_2} k(t,s_1)dt + \int_{s_2}^{x} k(t,s_1)dt - \int_{s_2}^{x} k(t,s_2)dt| \\
&\le \int_{s_1}^{s_2} |k(t,s_1)|dt + \int_{s_2}^{x} |(k(t,s_1)-k(t,s_2)|dt \\
&\le M|s_1-s_2| + L|s_1-s_2|(x-s_2) \\
&\le (M+L(b-a))|s_1-s_2|,
\end{aligned}$$

where $M = \max_{(x,s)\in G} |k(x,s)|$ and L is the Lipschitz constant of k and thus k_1 satisfies in Lipschitz condition. □

Lemma 3.2. *Let y be H-differentiable and k does not change sign. Then $k_1(x,s).y(s)$ is a Lipschitz function with respect to s.*

Proof. Since y is H-differentiable, it is a Lipschitz function i.e.

$$\exists\beta \ge 0, \qquad D(y(s_1),y(s_2)) \le \beta D(s_1,s_2).$$

Without loss of generality, we assume that k is positive. Then for $a \le s_1 \le s_2 \le x$ we have

$$\begin{aligned}
D(k_1(x,s_1).y(s_1),k_1(x,s_2).y(s_2)) &\le D(k_1(x,s_1).y(s_1),k_1(x,s_2).y(s_1)) \\
&\quad +D(k_1(x,s_2).y(s_1),k_1(x,s_2).y(s_2)) \\
&= \sup_{r\in[0,1]} \max\{|k_1(x,s_1)[y(s_1)]_-^r - k_1(x,s_2)[y(s_1)]_-^r| \\
&\qquad ,|k_1(x,s_1)[y(s_1)]_+^r - k_1(x,s_2)[y(s_1)]_+^r|\} \\
&\quad +|k_1(x,s_2)|D(y(s_1),y(s_2)) \\
&\le \sup_{r\in[0,1]} \max\{|[y(s_1)]_-^r||k_1(x,s_1)-k_1(x,s_2)| \\
&\qquad ,|[y(s_1)]_+^r||k_1(x,s_1)-k_1(x,s_2)|\} \\
&\quad +M_1 D(y(s_1),y(s_2)) \\
&\le \|y(s_1)\||k_1(x,s_1)-k_1(x,s_2)| + M_1\beta D(s_1,s_2) \\
&\le (M_2L_1+M_1\beta)D(s_1,s_2),
\end{aligned}$$

where $M_1 = \max_{(x,s)\in G} |k_1(x,s)|$, $M_2 = \max_{s\in I} \|y(s)\|$ and L_1 is the Lipschitz constant of $k_1(x,s)$ with respect to s. □

Lemma 3.3. *Let y be H-differentiable and k does not change sign. Then*
(i) *$\forall x \in I$, $k_1(x,s).y(s)$ is continuous with respect to s,*
(ii) *$\{k_1(x,s).y(s)|s \in I\}$ is an equicontinuous family of functions.*

Proof. It is an immediate consequence of the definitions of continuity and equicontinuity. □

3.1. **Existence Result.** Consider the space of functions

$$X = \{f : I \to \mathbb{R}_{\mathcal{F}} | f \textit{ is continuous}\}$$

along with the metric defined by

$$D^*(f, g) = \sup_{t \in I} D(f(t), g(t)).$$

It is worth to remind that (X, D^*) is a complete metric space (see e.g. [15]) with the norm defined by

$$\|f\|_X = D^*(f, \tilde{0}), \qquad \forall f \in X.$$

Theorem 3.4. *Let $k : G \to \mathbb{R}$ be a continuous function with no sign changes and $f : I \to \mathbb{R}_{\mathcal{F}}$ be continuous. Then problem (4) has a unique solution.*

Proof. Let the operator A be defined on X by

$$(Ay)(x) = g(x) + \int_a^x k_1(x, s).y(s)ds, \qquad \forall x \in I.$$

We claim that $A : X \to X$. To do this, it is evident that $(Ay)(x) \in \mathbb{R}_{\mathcal{F}}$ for all $x \in I$ and thus $Ay : I \to \mathbb{R}_{\mathcal{F}}$. It is sufficient to prove it's continuity. For this, let $t_1, t_2 \in I$, $t_1 \leq t_2$ and $y \in X$. Then

$$\begin{aligned}
D\left(Ay(t_1), Ay(t_2)\right) &= D\left(g(t_1) + \int_a^{t_1} k_1(t_1, s).y(s)ds, g(t_2) + \int_a^{t_2} k_1(t_2, s).y(s)ds\right) \\
&= D\left(g(t_1), g(t_2)\right) \\
&\quad + D\left(\int_a^{t_1} k_1(t_1, s).y(s)ds, \int_a^{t_1} k_1(t_2, s).y(s)ds + \int_{t_1}^{t_2} k_1(t_2, s).y(s)ds\right) \\
&\leq D\left(g(t_1), g(t_2)\right) + D\left(\int_a^{t_1} k_1(t_1, s).y(s)ds, \int_a^{t_1} k_1(t_2, s).y(s)ds\right) \\
&\quad + D\left(\tilde{0}, \int_{t_1}^{t_2} k_1(t_2, s).y(s)ds\right) \\
&\leq D\left(g(t_1), g(t_2)\right) + \int_a^{t_1} D(k_1(t_1, s).y(s), k_1(t_2, s).y(s))ds \\
&\quad + \int_{t_1}^{t_2} D(\tilde{0}, k_1(t_2, s).y(s))ds. \qquad (5)
\end{aligned}$$

Since $k_1(t_2, s).y(s)$ is continuous (see lemma 3.3), there exists $M > 0$ such that $\|k_1(t_2, s).y(s)\|_X \leq M$. Let $\epsilon > 0$ be arbitrary, then there are $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$ such that $\epsilon_1 + \epsilon_2(b - a) + \epsilon_3 M < \epsilon$. For $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that $D\left(g(t_1), g(t_2)\right) \leq \epsilon_1$ for $|t_1 - t_2| \leq \delta_1$. From lemma 3.3, $\{k_1(x, s).y(s) | s \in I\}$ is an equicontinuous family of functions, thus for $\epsilon_2 > 0$, there exists $\delta_2 > 0$ such that

$$\sup_{s \in I} D\left(k_1(t_1, s).y(s), k_1(t_2, s).y(s)\right) < \epsilon_2,$$

for $|t_1 - t_2| \leq \delta_2$. Let $\delta = \min\{\delta_1, \delta_2, \epsilon_3\}$ and $t_1, t_2 \in I$ with $|t_1 - t_2| \leq \delta$. Then from (5) we have

$$D\left(Ay(t_1), Ay(t_2)\right) \leq \epsilon_1 + \epsilon_2(b - a) + \epsilon_3 M < \epsilon.$$

To prove that A is a contraction mapping, we set

$$L_0 = \max_{(x,s)\in G} |k_1(x,s)|$$

and define

$$D_\beta(f,g) := \sup_{x\in I} e^{-\beta L_0 x} D(f(x),g(x)), \qquad \beta \geq 1$$

as a metric on X which is equivalent to the metric D^* (see e.g. [14]). Thus for $y_1, y_2 \in X$ and $t \in I$, we have

$$\begin{aligned}
D(Ay_1(t), Ay_2(t)) &= D\left(g(t) + \int_a^t k_1(t,s).y_1(s)ds, g(t) + \int_a^t k_1(t,s).y_2(s)ds\right) \\
&= D\left(\int_a^t k_1(t,s).y_1(s)ds, \int_a^t k_1(t,s).y_2(s)ds\right) \\
&\leq \int_a^t D(k_1(t,s).y_1(s), k_1(t,s).y_2(s))ds \\
&\leq \int_a^t L_0 D(y_1(s), y_2(s))ds \\
&= \int_a^t L_0 e^{\beta L_0 s} e^{-\beta L_0 s} D(y_1(s), y_2(s))ds \\
&\leq \int_a^t L_0 e^{\beta L_0 s} D_\beta(y_1, y_2)ds \\
&= D_\beta(y_1, y_2)\frac{e^{\beta L_0 t} - e^{\beta L_0 a}}{\beta} \\
&= D_\beta(y_1, y_2)e^{\beta L_0 t}\frac{1 - e^{\beta L_0 (a-t)}}{\beta},
\end{aligned}$$

which implies

$$\begin{aligned}
D(Ay_1(t), Ay_2(t))e^{-\beta L_0 t} &\leq D_\beta(y_1, y_2)\frac{1 - e^{\beta L_0 (a-t)}}{\beta} \\
&\leq D_\beta(y_1, y_2)\frac{1 - e^{\beta L_0 (a-b)}}{\beta}
\end{aligned}$$

and consequently

$$D_\beta(Ay_1, Ay_2) \leq D_\beta(y_1, y_2)\frac{1 - e^{\beta L_0 (a-b)}}{\beta}.$$

Since $\frac{1-e^{\beta L_0 (a-b)}}{\beta} < 1$, the operator A is a contraction mapping. By the Banach fixed point theorem we conclude that the initial value problem (3)-(4) has a unique solution. □

3.2. **Numerical Solution.** In order to get a particular method, we choose a uniform partition $\Delta : a = x_0 < x_1 < ... < x_n = b$ of the interval $[a,b]$ with step size $h = \frac{b-a}{n}$ such that all resulting quadrature rules involve only intervals of the form $[a, x_i]$ using just the points $x_0, x_1, ..., x_i$. Then we approximate the unknown function y at the points x_i, $\forall i = 0, ..., n$ as follows:

(i) $y_0 = y(a)$,

(ii) $y_i = g(x_i) + Q_i := g(x_i) + Q_{[a,x_i]}(k_1(x_i,.).\hat{y}(.)), \quad i = 1, 2, ...,$
where we define $\hat{y}(x_l) := y_l$ and

$$Q_i := Q_{[a,x_i]}(f) = \sum_{l=0}^{i} \omega_{l,i} f(x_l)$$

as a quadrature rule. We have explicit or implicit equation for y_i either $\omega_{i,i} = 0$ which corresponds to the left rectangular and hybrid quadrature rules or $\omega_{i,i} \neq 0$ which corresponds to the trapezoidal quadrature rule. In the rectangular rule, we do not need to solve any equation but the accuracy is low and its error is bounded by

$$\| R(Q_i) \| \leq \frac{L}{2} \sum_{j=1}^{i} (x_j - x_{j-1})^2 \leq \frac{L}{2} \sum_{j=1}^{n} (x_j - x_{j-1})^2 = \frac{L(b-a)}{2} h, \tag{6}$$

where $R(Q_i)$ is the quadrature error and L is a Lipschitz constant (see theorem 2.6). From lemma 3.2, $k_1(x,s).y(s)$ is Lipschitz with respect to s with a Lipschitz constant.

For more accurate results, we apply trapezoidal rule which leads to implicit method needing the equation

$$y_i = g(x_i) + \frac{h}{2}.k_1(x_i, x_0).y(x_0) + h.\sum_{l=1}^{i-1} k_1(x_i, x_l).y_l + \frac{h}{2}.k_1(x_i, x_i).y_i, \tag{7}$$

to be solved for $y_i, i = 1, 2, ..., n$.
It is required to verify the existence and uniqueness of solution of this equation. To this end we prove the following lemma.

Lemma 3.5. *Let the step size h be sufficiently small:*

$$h < 2/L_0.$$

Then equation (7) has the unique solution y_i.

Proof. To simplify notations, let

$$c_i := g(x_i) + \frac{h}{2}.k_1(x_i, x_0).y(x_0) + h.\sum_{l=1}^{i-1} k_1(x_i, x_l).y_l$$

and define the operator $T : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$T(\zeta) := c_i + \frac{h}{2}.k_1(x_i, x_i).\zeta.$$

Since $k_1(x_i, x_i).\zeta$ is a Lipschitz function with the Lipschitz constant L_0 with respect to ζ, we have

$$D(T(\zeta), T(\xi)) = \frac{h}{2} D(k_1(x_i, x_i).\zeta, k_1(x_i, x_i).\xi) \leq \frac{L_0 h}{2} D(\zeta, \xi)$$

and since $h < 2/L_0$, it follows that T is contraction in the complete metric space $(\mathbb{R}_{\mathcal{F}}, D)$. The Banach fixed point theorem can be applied to complete the proof. □

Furthermore, the Banach fixed point theorem offers an iterative method to obtain approximate solution of (7), i.e.

$$y_i^{(0)} = y_{i-1}, \quad y_i^{(m+1)} = c_i + \frac{h}{2}.k_1(x_i, x_i).y_i^{(m)}, \ m = 0, 1, 2, ... \tag{8}$$

For the trapezoidal quadrature rule, from (2), we have

$$\| R(Q_i) \| \leq \frac{L}{4} h(b-a).$$

To avoid solving any equation, we divide the interval $[a, x_i]$ into the intervals $[a, x_{i-1}]$ and $[x_{i-1}, x_i]$. For the first interval we use trapezoidal rule and for the second interval we use left-side rectangular. This quadrature rule is called a hybrid method. The resulting explicit equation is given by

$$y_i = g(x_i) + \frac{h}{2}.k_1(x_i, x_0).y(x_0) + h.\sum_{l=1}^{i-2} k_1(x_i, x_l).y_l + \frac{3h}{2}.k_1(x_i, x_{i-1}).y_{i-1},$$

$for\ i = 1, 2, ..., n$ with the upper error bound

$$\begin{aligned} \| R(Q_i) \| &= D\left(\int_a^{x_i} f(t)dt, \frac{h}{2}.f(x_0) + \sum_{l=1}^{i-2} h.f(x_l) + \frac{3h}{2}.f(x_{i-1}) \right) \\ &\leq D\left(\int_a^{x_{i-1}} f(t)dt, \frac{h}{2}.f(x_0) + \sum_{l=1}^{i-2} h.f(x_l) + \frac{h}{2}.f(x_{i-1}) \right) \\ &\quad + D\left(\int_{x_{i-1}}^{x_i} f(t)dt, h.f(x_{i-1}) \right), \end{aligned}$$

which is summarized by using (1) and (2) as

$$\| R(Q_i) \| \leq \frac{L(b-a)}{4} h + \frac{L}{2} h^2 = \frac{L(b-a)}{2} h(\frac{1}{2} + \frac{1}{n}).$$

Therefore for the given quadrature rules, we have

$$\| R(Q_i) \| \leq C_R h. \tag{9}$$

where C_R takes the values $\frac{L(b-a)}{2}$, $\frac{L(b-a)}{4}$ and $\frac{L(b-a)}{2}(\frac{1}{2}+\frac{1}{n})$ for left-side rectangular, trapezoidal and hybrid quadrature rules respectively.

3.3. **Error Estimation.** In this section, we discuss upper bound of the error function

$$d_i := D(y_i, y(x_i)),$$

where y_i is the approximate value of $y(x_i)$.

Lemma 3.6. (see e.g. [14]) *Assume that there are the numbers $\alpha, \beta_l \geq 0$ and $0 \leq M_0 \leq 1$ such that*

$$0 \leq d_i \leq \alpha + \sum_{l=0}^{i-1} \beta_l d_l + M_0 d_i \qquad i \geq 0.$$

Then for all $i \geq 0$, we have

$$d_i \leq \frac{\alpha}{1-M_0} \prod_{l=0}^{i-1} (1 + \frac{\beta_l}{1-M_0}) \leq \frac{\alpha}{1-M_0} \exp\left(\sum_{l=0}^{i-1} \frac{\beta_l}{1-M_0} \right).$$

Theorem 3.7. *(rectangular and hybrid quadrature rules) Assume that $k(x,t)$ is continuous and has no sign changes, $y(x)$ is H-differentiable and $f(x)$ is continuous. Then the error estimate*

$$d_i \leq Ch$$

holds for the approximation y_i defined in subsection 3.2 by left-side rectangular or hybrid quadrature rule with $C = C_R \exp(L_0 ih)$.

Proof. Due to (4) the initial error vanishes:

$$d_0 = 0$$

while for $i > 0$, we have

$$\begin{aligned} d_i &= D\left(g(x_i) + \sum_{l=0}^{i-1} \omega_{l,i}.k_1(x_i,x_l).y_l, g(x_i) + \int_a^x k_1(x_i,s).y(s)ds\right) \\ &= D\left(\sum_{l=0}^{i-1} \omega_{l,i}.k_1(x_i,x_l).y_l, \sum_{l=0}^{i-1} \omega_{l,i}.k_1(x_i,x_l).y(x_l) + R(Q_i)\right) \\ &\leq D\left(\sum_{l=0}^{i-1} \omega_{l,i}.k_1(x_i,x_l).y_l, \sum_{l=0}^{i-1} \omega_{l,i}.k_1(x_i,x_l).y(x_l)\right) + D(0, R(Q_i)) \\ &\leq \sum_{l=0}^{i-1} \mid \omega_{l,i} \mid D\left(k_1(x_i,x_l).y_l, k_1(x_i,x_l).y(x_l)\right) + \parallel R(Q_i) \parallel \\ &\leq \sum_{l=0}^{i-1} \mid \omega_{l,i} \mid L_0 D(y_l, y(x_l)) + \parallel R(Q_i) \parallel \\ &\leq \sum_{l=0}^{i-1} \mid \omega_{l,i} \mid L_0 d_l + \parallel R(Q_i) \parallel, \end{aligned} \tag{10}$$

where $R(Q_i)$ is the quadrature error for the integrand

$$k_i(x_i, s).y(s).$$

From (9) and (10), we finally obtain

$$d_i \leq L_0 \sum_{l=0}^{i-1} \mid \omega_{l,i} \mid d_l + C_R h. \tag{11}$$

To simplify the inequality (11), we introduce the new constants $\beta_l = L_0 \mid \omega_{l,i} \mid$, $\alpha = C_R h$ and $M_0 = 0$. Then (11) can be written as

$$0 \leq d_i \leq \alpha + \sum_{l=0}^{i-1} \beta_l d_l + M_0 d_i \qquad i > 0. \tag{12}$$

Consequently, we conclude from the lemma 3.6, equation (3.3) and $d_0 = 0$ that

$$d_i \leq \frac{\alpha}{1 - M_0} \exp\left(\sum_{l=0}^{i-1} \frac{\beta_l}{1 - M_0}\right).$$

Substituting $\alpha = C_R h$, $\beta_l = L_0 \mid \omega_{l,i} \mid$ and $M_0 = 0$, we obtain

$$d_i \leq C_R h \exp\left(\sum_{l=0}^{i-1} L_0 h\right) = C_R h \exp(L_0 i h),$$

which completes the proof. □

For the error estimation of trapezoidal quadrature rule we state the following theorem, since in this case the equation for y_i is different from two other cases.

Theorem 3.8. *(trapezoidal quadrature rule) Assume that $k(x,t)$ is continuous and does not change sign, $y(x)$ is H-differentiable and $f(x)$ is continuous. To get an approximation to the solution of equation(7) use the first iterate $\tilde{y}_i = y_i^{(1)}$ from (8). Then the following error estimate holds for $\tilde{y}_i$*

$$D(y(x_i), \tilde{y}_i) \leq Ch,$$

where $C = (L_y + C_R)exp(ihL_0)$.

Proof. The approximation $\tilde{y}_i$ of y_i is obtained

$$\tilde{y}_0 := y_0 = y(x_0), \quad y_i^{(0)} := \tilde{y}_{i-1}, \quad \tilde{y}_i := y_i^{(1)}$$

and so

$$\begin{aligned}
d_i &= D(\tilde{y}_i, y(x_i)) \\
&= D(g(x_i) + h.\sum_{j=0}^{i-1}{}' k_1(x_i,x_j).\tilde{y}_j + \frac{h}{2}.k_1(x_i,x_i).\tilde{y}_{i-1}, \\
&\quad g(x_i) + h.\sum_{j=0}^{i}{}'' k_1(x_i,x_j).y(x_j) + R(Q_i)) \\
&\leq h\sum_{j=0}^{i-1}{}' D\left(k_1(x_i,x_j).\tilde{y}_j, k_1(x_i,x_j).y(x_j)\right) \\
&\quad + \frac{h}{2} D(k_1(x_i,x_i).\tilde{y}_{i-1}, k_1(x_i,x_i).\tilde{y}(x_{i-1})) + D(\tilde{0}, R(Q_i)) \\
&\leq hL_0 \sum_{j=0}^{i-1}{}' D\left(\tilde{y}_j, y(x_j)\right) + \frac{hL_0}{2} D(\tilde{y}_{i-1}, y(x_i)) + \|R(Q_i)\| \\
&\leq hL_0 \sum_{j=0}^{i-1}{}' d_j + \frac{hL_0}{2} D(\tilde{y}_{i-1}, y(x_{i-1})) + D(y(x_{i-1}), y(x_i)) + C_R h \\
&\leq hL_0 \sum_{j=0}^{i-1}{}' d_j + \frac{hL_0}{2} d_{i-1} + L_y h + C_R h \\
&= hL_0 \sum_{j=0}^{i-2}{}' d_j + \frac{3hL_0}{2} d_{i-1} + (L_y + C_R)h.
\end{aligned}$$

Here, $\sum'$ denotes the sum with the weights 1 for $1 \leq j \leq i-1$ and $\frac{1}{2}$ for $j=0$ and $\sum''$ denotes the sum with the weights 1 for $1 \leq j \leq i-1$ and $\frac{1}{2}$ for $j=0,i$, and L_y is the Lipschitz constant of y (note that y is Lipschitz, since y is H-differentiable). Now lemma 3.6 is applicable with $M_0=0$, $\alpha=(L_y+C_R)h$ and $\sum_{j=0}^{i-1}\beta_j = ihL_0$ which yields the desired result. □

Example 3.9. Consider the fuzzy number A along with the r-cuts $[A]^r = [r^2 + r, 4-r^3-r]$ for $r \in [0,1]$. Let the functions $k : [0,1]\times[0,1] \to \mathbb{R}$ and $f : [0,1] \to \mathbb{R}_{\mathcal{F}}$ be given by

$$\begin{aligned} k(x,t) &= 0.1\sin(\frac{x}{2})\sin(t), \\ f(x) &= (\frac{1}{2}\cos(\frac{x}{2}) - 0.1\sin^2(\frac{x}{2}) + \frac{0.1}{3}\sin(\frac{x}{2})\sin(\frac{3x}{2})).A. \end{aligned}$$

Then the fuzzy integro-differential equation

$$\begin{aligned} & y'(x) = f(x) + \int_0^x k(x,t).y(t)dt, \qquad x \in [0, \frac{\pi}{2}], \\ & y(0) = 0 \end{aligned}$$

has the exact solution $y(x) = \sin(\frac{x}{2}).A$. The results based on rectangular, trapezoidal and hybrid quadrature rules for $n = 10$ are shown in Figures 1-3. In the figures, the points $(x_i, [y_i]^{r_j})$ are displayed for $r_j = \frac{j}{4},\ j = 0, ..., 4$ where $[y_i]^{r_j}$ is r_j-cut of the approximate solution y_i at the points $x_i = i\frac{\pi}{20}$ for $i = 0, ..., 10$. The lines show r-cuts of the exact solution for $r = 0, 1$. Numerical results for this example are reported in tables 1-3 for the rectangular, trapezoidal and hybrid methods respectively. In these tables, $E^r = \max\{|y_-^r(x_i) - {y_i}_-^r|, |y_+^r(x_i) - {y_i}_+^r|\}$. As you see, the results in Tables 1,2 are the same, since in converting the integro-differential equation to the corresponding integral equation the term $cos(\frac{s}{2}) - cos(\frac{x}{2})$ appears under integral sign and it is equal to zero at the points $x = x_i$, $i = 1, ..., n$ for the trapezoidal rule. The *CPU time* reported at the end of each table is the running time of the program for getting all the results in that table.

4. Conclusions and Further Research

Approximation of the solution of linear fuzzy integro-differential equation in discrete form was studied and error estimates was obtained. We used the left rectangular and trapezoidal quadrature rules. First of them leads to an explicit method and the second one leads to an implicit and more accurate method. Finally we combined two mentioned quadrature rules to construct an explicit and accurate method. It is worth to note that, although the hybrid method is more time consuming but enjoys more accurate results. We have used H-differentiability which is (i)-differentiability defined in [8]. For the future works we will use (ii) or (iii) or (iv) differentiability and also discuss solvability of nonlinear fuzzy integro-differential equations.

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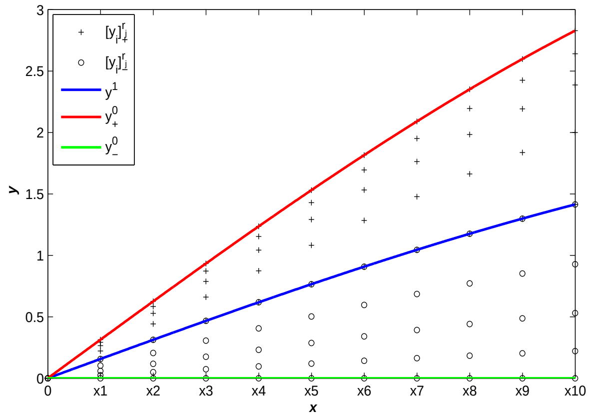

FIGURE 1. Discrete Approximation of the Solution by Rectangular Quadrature Rule

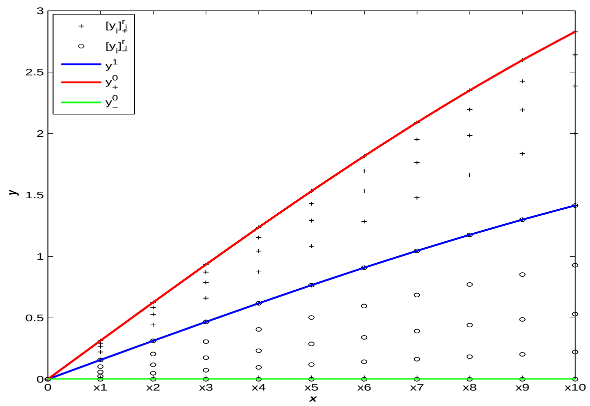

FIGURE 2. Discrete Approximation of the Solution by Trapezoidal Quadrature Rule

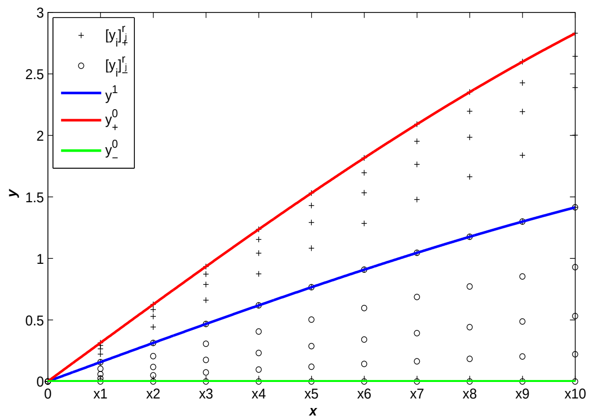

FIGURE 3. Discrete Approximation of the Solution by Hybrid Quadrature Rule

x	E^0	$E^{1/4}$	$E^{1/2}$	$E^{3/4}$	E^1
$\pi/20$	.6357e-6	.5935e-6	.5363e-6	.4494e-6	.3178e-6
$2\pi/20$	.5929e-5	.5535e-5	.5002e-5	.4192e-5	.2964e-5
$3\pi/20$	.1996e-4	.1864e-4	.1685e-4	.1412e-4	.9982e-5
$4\pi/20$	.4575e-4	.4271e-4	.3860e-4	.3235e-4	.2287e-4
$5\pi/20$	.8481e-4	.7918e-4	.7156e-4	.5997e-4	.4241e-4
$6\pi/20$	.1370e-3	.1279e-3	.1156e-3	.9685e-4	.6849e-4
$7\pi/20$	.2003e-3	.1870e-3	.1690e-3	.1416e-3	.10028e-3
$8\pi/20$	.2712e-3	.2532e-3	.2289e-3	.1918e-3	.1356e-3
$9\pi/20$	.3449e-3	.3219e-3	.2910e-3	.2438e-3	.1724e-3
$\pi/2$	.4153e-3	.3878e-3	.3504e-3	.2937e-3	.2077e-3
CPU time:	2.408″				

Table 1. Numerical Results Based on Rectangular Rule

x	E^0	$E^{1/4}$	$E^{1/2}$	$E^{3/4}$	E^1
$\pi/20$	.6357e-6	.5935e-6	.5363e-6	.4494e-6	.3178e-6
$2\pi/20$	.5929e-5	.5535e-5	.5002e-5	.4192e-5	.2964e-5
$3\pi/20$	.1996e-4	.1864e-4	.1685e-4	.1412e-4	.9982e-5
$4\pi/20$	.4575e-4	.4271e-4	.3860e-4	.3235e-4	.2287e-4
$5\pi/20$	.8481e-4	.7918e-4	.7156e-4	.5997e-4	.4241e-4
$6\pi/20$	.1370e-3	.1279e-3	.1156e-3	.9685e-4	.6849e-4
$7\pi/20$	.2003e-3	.1870e-3	.1690e-3	.1416e-3	.1003e-3
$8\pi/20$	.2712e-3	.2532e-3	.2289e-3	.1918e-3	.1356e-3
$9\pi/20$	.3448e-3	.3219e-3	.2910e-3	.2438e-3	.1724e-3
$\pi/2$	.4153e-3	.3878e-3	.3504e-3	.2937e-3	.2077e-3
CPU time:	2.437″				

Table 2. Numerical Results Based on Trapezoidal Rule

x	E^0	$E^{1/4}$	$E^{1/2}$	$E^{3/4}$	E^1
$\pi/20$	.6357e-6	.5935e-6	.5363e-6	.4494e-6	.3178e-6
$2\pi/20$	.1189e-5	.1110e-5	.1003e-5	.8404e-6	.5943e-6
$3\pi/20$	.2656e-4	.2480e-4	.2241e-4	.1878e-4	.1328e-4
$4\pi/20$	.9620e-4	.8981e-4	.8118e-4	.6801e-4	.4810e-4
$5\pi/20$	.2255e-3	.2105e-3	.1902e-3	.1594e-3	.1127e-3
$6\pi/20$	.4225e-3	.3944e-3	.3564e-3	.2987e-3	.2112e-3
$7\pi/20$	.6867e-3	.6411e-3	.5794e-3	.4855e-3	.3433e-3
$8\pi/20$	.1009e-2	.9418e-3	.8511e-3	.7132e-3	.5044e-3
$9\pi/20$	.1371e-2	.1280e-2	.1159e-2	.9694e-3	.6856e-3
$\pi/2$	.1749e-2	.1633e-2	.1476e-2	.1237e-2	.8747e-3
CPU time:	8.749″				

Table 3. Numerical Results Based on Hybrid Rule

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SET-NORM EXHAUSTIVE SET MULTIFUNCTIONS

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Abstract. In this paper we present some properties of set-norm exhaustive set multifunctions and also of atoms and pseudo-atoms of set multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity.

1. **Introduction**

The subject of the present paper concerns fuzzy set multifunctions. Non-additive set functions and fuzzy sets have been intensively studied by many authors (e.g., Asahina [1], Choquet [4], Daneshgar and Hashemi [7], Denneberg [9], Drewnowski [10], Dubois and Prade [11], Funiokova [12], Li [16], Merghadi and Aliouche [17], Pap [18], Precupanu [19], Shafer [20], Sugeno [21], Suzuki [22], Zadeh [23], Vaezpour and Karini [24], Wen, Shi and Li [25], Wu and Bo [26]), due to its applications in statistics, economy, theory of games, human decision making.

In our previous papers [5,6,13-15] we extended and studied different concepts (such as pseudo-atom, Darboux property, continuity, exhaustivity, regularity) to the set-valued case.

In [5] we introduced the notion of set-norm on the family of non-empty subsets of a real linear space and studied different notions of continuous set multifunctions with respect to a set-norm.

This paper contains three sections. In the second section, properties of set-norm exhaustive set multifunctions are presented. These set multifunctions (such as probability multimeasures) are used in control, robotics, decision theory (in Bayesian estimation) or in statistical inference (Dempster [8]). In the third section, we present different properties of atoms and pseudo-atoms of set multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity. Our results generalize to set-valued case important problems in measure theory, such as non-atomicity or pure atomicity (Aumann and Shapley [2]), that have applications in coincidence and rigidity phenomena (Chiţescu [3]).

2. **Non-Additive Set Multifunctions**

Let T be an abstract nonvoid set, $\mathcal{C}$ a ring of subsets of T and $\mathcal{P}(T)$ the family of all subsets of T.

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In the sequel, $(X, +, 0)$ will be a commutative semigroup with unity 0 and $\mathcal{P}_0(X)$ the family of non-empty subsets of X. On $\mathcal{P}_0(X)$ we consider an order relation denoted by " $\leq$ ". We write $E < F$ if $E \leq F$ and $E \neq F$, for every $E, F \in \mathcal{P}_0(X)$. The notation $F \geq E$ ($F > E$ respectively) will often be used in the place of $E \leq F$ ($E < F$ respectively). We shall write $(\mathcal{P}_0(X), \leq)$. For every $E, F \in \mathcal{P}_0(X)$, let $E + F = \{x + y | x \in E, y \in F\}$.

Example 2.1. I. The usual set inclusion "$\subseteq$" is an order relation on $\mathcal{P}_0(X)$.

If X is a normed space, then $\mathcal{P}_c(X)$ is the family of non-empty closed subsets of X and $\mathcal{P}_{cb}(X)$ is the family of non-empty closed bounded subsets of X.

The set of all real numbers is denoted by $\mathbb{R}$. We denote $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$, where $\mathbb{N}$ is the set of all positive integers.

Definition 2.2. [5] A function $|\cdot| : \mathcal{P}_0(X) \to [0, +\infty]$ is called *a set-norm* on $\mathcal{P}_0(X)$ if it satisfies the conditions:

(i) $|E| = 0 \Leftrightarrow E = \{0\}, \forall E \in \mathcal{P}_0(X)$.

(ii) $|E + F| \leq |E| + |F|, \forall E, F \in \mathcal{P}_0(X)$.

Definition 2.3. [5] A set-norm $|\cdot|$ on $(\mathcal{P}_0(X), \leq)$ is called *monotone* if for every sets $E, F \in \mathcal{P}_0(X)$, $E \leq F \Rightarrow |E| \leq |F|$. We denote $(\mathcal{P}_0(X), \leq, |\cdot|)$ when $(\mathcal{P}_0(X), \leq)$ is endowed with a monotone set-norm $|\cdot|$.

Example 2.4. Let $(X, \|\cdot\|)$ be a real normed space and $|E|_s = \sup\limits_{x \in E} \|x\|$, for every $E \in \mathcal{P}_0(X)$. Then the function $|\cdot|_s$ is a monotone set-norm on $(\mathcal{P}_0(X), \subseteq)$ and we denote this by $(\mathcal{P}_0(X), \subseteq, |\cdot|_s)$.

Definition 2.5. A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is called:

(i) a *multimeasure* if $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) = \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

(ii) *null-additive* if for every $A, B \in \mathcal{C}$,

$$\mu(B) = \mu(\emptyset) \Rightarrow \mu(A \cup B) = \mu(A).$$

(iii) *null-null-additive* if for every $A, B \in \mathcal{C}$,

$$\mu(A) = \mu(B) = \mu(\emptyset) \Rightarrow \mu(A \cup B) = \mu(\emptyset).$$

Definition 2.6. Let $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)$ be a set multifunction. μ is said to be:

(i) *monotone* if $\mu(A) \leq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

(ii) *fuzzy* if μ is monotone and $\mu(\emptyset) = \{0\}$.

(iii) *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$.

(iv) *a multisubmeasure* if μ is fuzzy and subadditive.

Remark 2.7. I. If X is a normed space and μ is $\mathcal{P}_c(X)$-valued, then in the definition of a multi(sub)measure, it usually appears " $\overset{\bullet}{+}$ " instead of " $+$ ", because the sum of two closed sets is not always closed.

II. The following implications hold:

(i) If μ is a multisubmeasure, then μ is null-additive.

(ii) If μ is null-additive, then μ is null-null-additive.

III. The concepts in Definitions 2.5 and 2.6 do not reduce to the usual single-valued case. The difficulty arises here since we have to consider an order relation on $\mathcal{P}_0(X)$ and many classical measure theory proof methods fail. For instance, if $\mu : (\mathcal{P}_0(\mathbb{R}), \subseteq)$ is single-valued and monotone, then μ reduces in fact to a constant function $\mu(A) = \{\mu(\emptyset)\}, \ \forall A \in \mathcal{C}$.

Moreover, $\mathcal{P}_0(X)$ (and also $\mathcal{P}_c(X)$) is not a linear space since $\mathcal{P}_0(X)$ is not a group with respect to the addition "+" defined by $M + N = \{x + y | x \in M, y \in N\}$, for every $M, N \in \mathcal{P}_0(X)$.

Definition 2.8. A set multifunction $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is said to be:

(i) *set-norm exhaustive* (shortly, *sn-exhaustive*) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every pairwise disjoint sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$.

(ii) *set-norm continuous* (shortly, *sn-continuous*) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, such that $A_n \searrow \emptyset$ (i.e. $A_n \supseteq A_{n+1}, \forall n \in \mathbb{N}^* \wedge \bigcap\limits_{n=1}^{\infty} A_n = \emptyset$).

(iii) *strongly-set-norm continuous* (shortly, *strongly sn-continuous*) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$ for every sequence $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ such that $A_{n+1} \subseteq A_n$, $\forall n \in \mathbb{N}^*$ and $\mu(\bigcap\limits_{n=1}^{\infty} A_n) = \{0\}$.

(iv) *null-continuous* if $\mu(\bigcup\limits_{n=1}^{\infty} A_n) = \{0\}$ for every sequence $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ such that $A_n \subseteq A_{n+1}$ and $\mu(A_n) = \{0\}$, for every $n \in \mathbb{N}^*$.

We now establish some relationships among the set multifunctions introduced in Definition 2.8.

We recall that $\mathcal{C}$ is a σ-ring if the following conditions hold:
(i) $A \backslash B \in \mathcal{C}$, for every $A, B \in \mathcal{C}$,
(ii) $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$, for every $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$.

Theorem 2.9. *If $\mathcal{C}$ is a σ-ring and $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is fuzzy and sn-continuous, then μ is sn-exhaustive.*

Proof. Let $(A_n)_{n\in\mathbb{N}^*}$ be a sequence of mutually disjoint sets of $\mathcal{C}$ and let $B_n = \bigcup\limits_{k=n}^{\infty} A_k$, for all $n \in \mathbb{N}^*$. Then $B_n \in \mathcal{C}$, for every $n \in \mathbb{N}^*$ and $B_n \searrow \emptyset$. Since μ is sn-continuous, it results $|\mu(B_n)| \to 0$, which implies $|\mu(A_n)| \to 0$. So μ is sn-exhaustive. □

Theorem 2.10. *Suppose $\mathcal{C}$ is a σ-ring and $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is fuzzy. If μ is null-null-additive and strongly-sn-continuous, then μ is null-continuous.*

Proof. Let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ such that $A_n \subseteq A_{n+1}$ and $\mu(A_n) = \{0\}$, for every $n \in \mathbb{N}^*$. Denote $A = \bigcup\limits_{n=1}^{\infty} A_n$. We recurrently define a subsequence (A_{n_k}) of (A_n) as follows.

Let $n_1 = 1$. For every $k \in \mathbb{N}^*$, since $\mu(A_{n_k}) = \{0\}$ and $A_{n_k} \cup (A \backslash A_n) \searrow A_{n_k}$ when $n \to \infty$, by the fact that μ is strongly-sn-continuous, we can choose n_{k+1} so that $n_{k+1} > n_k$ and

$$|\mu(A_{n_k} \cup (A \backslash A_{n_{k+1}}))| < \frac{1}{k}.$$

Denote $B = \bigcup_{k=1}^{\infty} (A_{n_{2k}} \backslash A_{n_{2k-1}})$ and $C = A \backslash B = A_{n_1} \cup \bigcup_{k=1}^{\infty} (A_{n_{2k+1}} \backslash A_{n_{2k}})$. For all $k \in \mathbb{N}^*$, since $B \subseteq A_{n_{2k}} \cup (A \backslash A_{n_{2k+1}})$, it results:

$$|\mu(B)| \leq |\mu(A_{n_{2k}} \cup (A \backslash A_{n_{2k+1}}))| < \frac{1}{2k}.$$

It follows $\mu(B) = \{0\}$. Analogously, for every $k \in \mathbb{N}^*$, since $C \subseteq A_{n_{2k-1}} \cup (A \backslash A_{n_{2k}})$, it follows that:

$$|\mu(C)| \leq |\mu(A_{n_{2k-1}} \cup (A \backslash A_{n_{2k}}))| < \frac{1}{2k-1},$$

which implies that $\mu(C) = \{0\}$. Since μ is null-null-additive, we obtain $\mu(A) = \mu(B \cup C) = \{0\}$. So, μ is null-continuous. □

Example 2.11. I. If $\mathcal{C}$ is finite, then every set multifunction $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$, with $\mu(\emptyset) = \{0\}$, is sn-exhaustive and sn-continuous.

II. Let $T = \mathbb{R}$, $\mathcal{C} = \{A \subseteq T | A \text{ is finite}\}$ and $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq, |\cdot|_s)$ be defined by $\mu(A) = [0, \nu(A)]$, where $\nu(A) = \begin{cases} 0, & A = \emptyset \\ 1 + \text{card}\, A, & A \neq \emptyset \end{cases}$, $\forall A \in \mathcal{C}$ and card$A$ is the number of elements in A. Then μ is sn-continuous, but not sn-exhaustive.

III. Let $T = \mathbb{N}$, $\mathcal{C} = \mathcal{P}(\mathbb{N})$ and $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq, |\cdot|_s)$ be defined by

$$\mu(A) = \begin{cases} \{0\}, & A \neq \mathbb{N} \\ \{0, 1\} \cup [3, 7], & A = \mathbb{N}. \end{cases}$$

μ is not null-null-additive, because there exist $A = \{0\}$ and $B = \mathbb{N}^*$ so that $\mu(A) = \mu(B) = \{0\}$, but $\mu(A \cup B) = \mu(\mathbb{N}) \neq \{0\}$.
μ is not null-continuous since there is $(A_n)_{n \in \mathbb{N}}$, $A_n = \{0, 1, 2, \ldots, n\}$, for every $n \in \mathbb{N}$, such that $A_n \subseteq A_{n+1}$ and $\mu(A_n) = \{0\}$, for all $n \in \mathbb{N}$, but $\mu(\bigcup_{n=0}^{\infty} A_n) = \mu(\mathbb{N}) \neq \{0\}$.

3. **Atoms and Pseudo-Atoms**

We now give some properties regarding atoms and pseudo-atoms of set multifunctions taking values in the family of nonvoid subsets of a commutative semigroup with unity.
In the sequel, $(X, +, 0)$ is a commutative semigroup with unity 0 and "$\leq$" is an order relation on $\mathcal{P}_0(X)$.

Definition 3.1. Let $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)$ be a set multifunction.

(i) A set $A \in \mathcal{C}$ is said to be *an atom* of μ if $\mu(A) > \mu(\emptyset)$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \mu(\emptyset)$ or $\mu(A \backslash B) = \mu(\emptyset)$.

(ii) A set $A \in \mathcal{C}$ is called *a pseudo-atom* of μ if $\mu(A) > \mu(\emptyset)$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \mu(\emptyset)$ or $\mu(B) = \mu(A)$.

(iii) μ is called *non-atomic* (*non-pseudo-atomic* respectively) if it has no atoms (pseudo-atoms respectively).

Remark 3.2. If μ is fuzzy, then μ is non-atomic (non-pseudo-atomic respectively) if and only if for every $A \in \mathcal{C}$ with $\mu(A) > \{0\}$, there exists $B \in \mathcal{C}$, so that $B \subseteq A$, $\mu(B) > \{0\}$ and $\mu(A \backslash B) > \{0\}$ ($\mu(B) < \mu(A)$ respectively).

Proposition 3.3. *Suppose $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)$ is fuzzy and null-additive (or a multimeasure). Then every atom of μ is a pseudo-atom of μ.*

Proof. (i) Suppose μ is fuzzy and null-additive and let $A \in \mathcal{C}$ be an atom of μ. Let $B \in \mathcal{C}$, $B \subseteq A$ so that $\mu(B) \neq \{0\}$. Since A is an atom of μ, it results $\mu(A \backslash B) = \{0\}$. Since μ is null-additive, it follows $\mu(A) = \mu(B \cup (A \backslash B)) = \mu(B)$ which proves that A is a pseudo-atom of μ.

(ii) Suppose μ is a multimeasure and let $A \in \mathcal{C}$ be an atom of μ. Let $B \in \mathcal{C}$, $B \subseteq A$ so that $\mu(B) \neq \{0\}$. Since A is an atom of μ, it results in $\mu(A \backslash B) = \{0\}$. Since μ is a multimeasure, we have:

$$\mu(A) = \mu(B \cup (A \backslash B)) = \mu(B) + \mu(A \backslash B) = \mu(B) + \{0\} = \mu(B).$$

So A is pseudo-atom of μ. □

Remark 3.4. The converse of Proposition 3.3 is not valid (see Example 3.7). As we shall see in the sequel, if X is a normed space, then this converse is true. To prove this we need the following lemma.

Lemma 3.5. *Let $(X, \|\cdot\|)$ be a normed-space and $A, B \in \mathcal{P}_0(X)$ so that $A + B = B$ and B is bounded. Then $A = \{0\}$.*

Proof. Since B is bounded, there is $M > 0$ so that $\|y\| \leq M$, for every $y \in B$. Let $a \in A$, $b \in B$. It results in $a + b \in A + B = B$. Then $2a + b = a + (a + b) \in A + B = B$. By induction we obtain that $na + b \in B$, for every $n \in \mathbb{N}$. It follows $\|na + b\| \leq M$. Consequently, we have $\|a\| \leq \frac{2M}{n}$, for every $n \in \mathbb{N}^*$, which proves that $a = 0$. So $A = \{0\}$. □

Proposition 3.6. *If X is a normed space and $\mu : \mathcal{C} \to (\mathcal{P}_b(X), \leq)$ is a multimeasure, then $A \in \mathcal{C}$ is a pseudo-atom of μ if and only if A is an atom of μ.*

Proof. I. Suppose A is an atom of μ and let $B \in \mathcal{C}$, $B \subseteq A$ such that $\mu(B) \neq \{0\}$. Since A is an atom of μ, it results in $\mu(A \backslash B) = \{0\}$. Since μ is a multimeasure, we have:

$$\mu(A) = \mu(A \backslash B) \cup B = \mu(A \backslash B) + \mu(B) = \{0\} + \mu(B) = \mu(B),$$

which proves that A is a pseudo-atom of μ.

II. Suppose A is a pseudo-atom of μ and let $B \in \mathcal{C}$, $B \subseteq A$ such that $\mu(B) \neq \{0\}$. Since A is a pseudo-atom of μ, we have $\mu(B) = \mu(A)$. Since μ is a multimeasure, it results in:

$$\mu(A) = \mu((A \backslash B) \cup B) = \mu(A \backslash B) + \mu(B) = \mu(A \backslash B) + \mu(A).$$

According to Lemma 3.5, it follows $\mu(A\backslash B) = \{0\}$. Consequently, A is an atom of μ. $\square$

Example 3.7. I. Let $T = \{a, b, c\}$, $\mathcal{C} = \mathcal{P}(T)$, $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ be defined for every $A \in \mathcal{C}$ by $\mu(A) = \begin{cases} [0,1], & \text{if } A \neq \emptyset, \\ \{0\}, & \text{if } A = \emptyset. \end{cases}$
Then μ is null-additive, $A = \{a, b\}$ is a pseudo-atom of μ, but not an atom of μ.

II. Let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$, $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ be defined for every $A \in \mathcal{C}$ by $\mu(A) = \begin{cases} [0,2]\,, & \text{if } A = T \\ [0,1]\,, & \text{if } A = \{b\} \\ \{0\}, & \text{if } A = \emptyset \text{ or } A = \{a\}. \end{cases}$
Then μ is not null-additive, T is an atom of μ, but not a pseudo-atom of μ.

III. Let $T = 2\mathbb{N} = \{0, 2, 4, \ldots\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \subseteq)$ be defined for every $A \in \mathcal{C}$ by

$$\mu(A) = \begin{cases} \{0\}, & \text{if } A = \emptyset \\ \frac{1}{2}A \cup \{0\}, & \text{if } A \neq \emptyset \end{cases}$$

where $\frac{1}{2}A = \{\frac{x}{2} |\, x \in A\}$. Then μ is a multisubmeasure.

If $A \in \mathcal{C}$ has $\text{card}A = 1$ and $A \neq \{0\}$ or $A \in \mathcal{C}, A = \{0, 2n\}$, $n \in \mathbb{N}^*$, then A is an atom of μ (and a pseudo-atom of μ too).

If $A \in \mathcal{C}$ has $\text{card}A \geq 2$ and there exist $a, b \in A$ such that $a \neq b$ and $ab \neq 0$, then A is not a pseudo-atom of μ (and not an atom of μ).

IV. Let $T = \mathbb{N}$, $\mathcal{C} = \mathcal{P}(\mathbb{N})$ and $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ be defined for every $A \in \mathcal{C}$ by

$$\mu(A) = \begin{cases} \{0\}, & \text{if } A \text{ is finite} \\ \{0\} \cup [n_A, +\infty), & \text{if } A \text{ is infinite and} \\ & n_A = \min A. \end{cases}$$

Then μ is monotone and non-pseudo-atomic.

From definitions we obtain the following properties of pseudo-atoms.

Proposition 3.8. *Let $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)$ be a fuzzy set multifunction.*

I. If $A \in \mathcal{C}$ is a pseudo-atom of μ and $B \in \mathcal{C}, B \subseteq A$ is so that $\mu(B) > \{0\}$, then B is a pseudo-atom of μ and $\mu(B) = \mu(A)$.

II. If $A, B \in \mathcal{C}$ are pseudo-atoms of μ and $\mu(A) \neq \mu(B)$, then $\mu(A \cap B) = \{0\}$.

III. Moreover, suppose μ is null-null-additive and let $A, B \in \mathcal{C}$ be pseudo-atoms of μ. Then the following statements hold:

(i) *If $\mu(A \cap B) = \{0\}$, then $A\backslash B$ and $B\backslash A$ are pseudo-atoms of μ and $\mu(A\backslash B) = \mu(A)$, $\mu(B\backslash A) = \mu(B)$.*
(ii) *If $\mu(A \cap B) > \{0\}$ and $\mu(A\backslash B) = \mu(B\backslash A) = \{0\}$, then $A \cap B$ is a pseudo-atom of μ and $\mu(A \triangle B) = \{0\}$ (where $A \triangle B = (A\backslash B) \cup (B\backslash A)$).*

Proof. I. Since $\mu(B) > \{0\}$ and A is a pseudo-atom of μ, it results in $\mu(B) = \mu(A)$. Let $C \in \mathcal{C}$, $C \subseteq B$ and suppose $\mu(C) > \{0\}$. Since $C \subseteq A$ and A is a pseudo-atom, it follows $\mu(C) = \mu(A)$. This shows that $\mu(C) = \mu(B)$. So B is a pseudo-atom of μ.

II. Let $a, B \in \mathcal{C}$ be pseudo-atoms of μ such that $\mu(A) \neq \mu(B)$. Suppose, by contrary, that $\mu(A \cap B) > \{0\}$. Since $A \cap B \subseteq A$, $A \cap B \subseteq B$ and A, B are pseudo-atoms of μ, according to Proposition 3.8-I, we have $\mu(A \cap B) = \mu(A)$ and $\mu(A \cap B) = \mu(B)$. It follows $\mu(A) = \mu(B)$, false!

III. (i) Suppose $\mu(A \backslash B) = \{0\}$. Since $\mu(A \cap B) = \{0\}$ and μ is null-null-additive, it results $\mu((A \cup B) \cup (A \backslash B)) = \mu(A) = \{0\}$, that is false because A is a pseudo-atom. So $\mu(A \backslash B) > \{0\}$. By Proposition 3.8-I, it follows that $A \backslash B$ is a pseudo-atom and $\mu(A \backslash B) = \mu(A)$. In the same way it follows that $B \backslash A$ is a pseudo-atom and $\mu(B \backslash A) = \mu(B)$.

(ii) Since $A \cap B \subseteq A$ and $\mu(A \cap B) > \{0\}$, by Proposition 3.8-I, it results that $A \cap B$ is a pseudo-atom of μ. Since μ is null-null-additive, we have $\mu(A \triangle B) = \{0\}$. □

Proposition 3.9. *Suppose $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)$ is a null-additive fuzzy set multifunction and $A \in \mathcal{C}$ is an atom (pseudo-atom respectively). If $E \in \mathcal{C}$ is so that $\mu(E) = \{0\}$, then $B = A \cup E$ is also an atom (pseudo-atom respectively).*

Proof. Suppose A is a pseudo-atom of μ and let $C \in \mathcal{C}$, $C \subseteq B = A \cup E$. We can set $C = (C \cap A) \cup (C \cap E)$. By the monotonicity of μ, we have $\mu(C \cap E) = \{0\}$. Since μ is null-additive, the following relation holds:

$$\mu(C \cap A) = \mu(C). \tag{1}$$

But A is a pseudo-atom of μ and $C \cap A \subseteq A$. Then we have $\mu(C \cap A) = \{0\}$ or $\mu(C \cap A) = \mu(A)$.

(i) If $\mu(C \cap A) = \{0\}$, by (1) it results $\mu(C) = \{0\}$.

(ii) If $\mu(C \cap A) = \mu(A)$, by (1) it results $\mu(A) = \mu(C)$. Since $\mu(E) = \{0\}$ and μ is null-additive, we have $\mu(B) = \mu(A)$. So $\mu(C) = \mu(B)$.
This shows that B is a pseudo-atom of μ.
The case when A is an atom of μ analogously follows. □

Proposition 3.10. *Suppose $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is a null-additive fuzzy set multifunction and let $A \in \mathcal{C}$ be an atom of μ. If $\{B_i\}_{i=1}^{n} \subset \mathcal{C}$ is a partition of A, then there exists a unique $i_0 \in \{1, \ldots, n\}$ such that $\mu(B_{i_0}) = \mu(A)$ and $\mu(B_i) = \{0\}$, for $i \in \{1, \ldots, n\}$, $i \neq i_0$.*

Proof. We have two cases:

I. $\mu(B_i) = \{0\}$, for every $i \in \{1, \ldots, n\}$. From the null-additivity of μ, it results $\mu(A) = \{0\}$, which is false.

II. There exists $i_0 \in \{1, \ldots, n\}$ so that $\mu(B_{i_0}) > \{0\}$.
Suppose without loss in generality that $\mu(B_1) > \{0\}$ and $\mu(B_2) > \{0\}$. Since A is an atom of μ, it follows $\mu(A \backslash B_1) = \{0\}$. Since $B_2 \subseteq A \backslash B_1$ and μ is fuzzy, $\mu(B_2) = \{0\}$, which is false.

It results that there exists a unique $i_0 \in \{1, \ldots, n\}$ so that $\mu(B_{i_0}) > \{0\}$. Since A is an atom of μ, it follows $\mu(A \backslash B_{i_0}) = \{0\}$. From the null-additivity of μ we obtain $\mu(A) = \mu(B_{i_0})$. Since $B_i \subseteq A \backslash B_{i_0}$, for every $i \in \{1, \ldots, n\} \backslash \{i_0\}$ and μ is fuzzy, it follows that $\mu(B_i) = \{0\}$, for every $i \in \{1, \ldots, n\} \backslash \{i_0\}$ and the proof is finished. □

Definition 3.11. For a set multifunction $\mu : \mathcal{C} \to (\mathcal{P}_0(X), |\cdot|)$, the following set function (called the variation of μ) is introduced:

$\overline{\mu} : \mathcal{P}(T) \to [0, +\infty]$, $\overline{\mu}(E) = \sup\{\sum_{i=1}^{n} |\mu(A_i)|; A_i \subseteq E, A_i \in \mathcal{C}, A_i \cap A_j = \emptyset, i \neq j, i, j \in \{1, \ldots, n\}, n \in \mathbb{N}^*\}$, for every $E \in \mathcal{P}(T)$.

Remark 3.12. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction. Then the following statements hold:

I. $|\mu(A)| \leq \overline{\mu}(A)$, for every $A \in \mathcal{C}$.

The inequality may be strict (see Example 3.13-II). It becomes an equality in supplementary hypothesis (see Proposition 3.14).

II. $\overline{\mu}(A) = 0 \Rightarrow |\mu(A)| = 0$, for every $A \in \mathcal{C}$.

III. Moreover, if $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is fuzzy, then we also have:

$$|\mu(A)| = 0 \Rightarrow \overline{\mu}(A) = 0, \quad \forall A \in \mathcal{C}. \tag{2}$$

Indeed, let $\{B_i\}_{i=1}^{n} \subset \mathcal{C}$ be a partition of A. Since μ is fuzzy, $\sum_{i=1}^{n} |\mu(B_i)| = 0$ and so, $\overline{\mu}(A) = 0$.

If μ is not fuzzy, then (2) may be false as we can see in Example 3.13-I.

Example 3.13. I. Let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq, |\cdot|_h)$ be defined by:

$$\mu(A) = \begin{cases} \{0\}, & A = \emptyset \text{ or } A = T \\ [0,1] \cup \{2,3\}, & A = \{a\} \text{ or } A = \{b\}. \end{cases}$$

We have $|\mu(T)| = 0$ and $\overline{\mu}(T) = 6$.

II. Let $T = \{1, 2, 3\}$, $\mathcal{C} = \mathcal{P}(T)$, $X = \{f | f : [0, +\infty) \to [0, +\infty)\}$ and $|E| = \sup_{f \in E} \|f\|_u$, for every $E \in \mathcal{P}_0(X)$, where $\|f\|_u = \sup_{x \in E} |f(x)|$. Let $\mu : \mathcal{C} \to (\mathcal{P}_0(X), |\cdot|)$ be defined by

$$\mu(A) = \{\mathcal{X}_{[0,n]} | n \in A\}, \quad \forall A \in \mathcal{C},$$

where $\chi_{[0,n]}$ is the characteristic function of $[0, n]$.

In this setting we have $|\mu(T)| = 3 < 6 = \overline{\mu}(T)$.

Proposition 3.14. *If $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is a fuzzy set multifunction and $A \in \mathcal{C}$ is an atom of μ, then $\overline{\mu}(A) = |\mu(A)|$.*

Proof. According to the Remark 3.12-I, we only have to prove:

$$\overline{\mu}(A) \leq |\mu(A)|. \tag{3}$$

Let $\{B_i\}_{i=1}^n \subset \mathcal{C}$ be an arbitrary partition of A, where $n \in \mathbb{N}^*$. We have two cases:

I. $\mu(B_i) = \{0\}$, for every $i \in \{1, \ldots, n\}$. Then $\sum\limits_{i=1}^{n} |\mu(B_i)| = 0 \leq |\mu(A)|$.

II. There exists $i_0 \in \{1, \ldots, n\}$ so that $\mu(B_{i_0}) > \{0\}$.

Suppose without loss of generality that $\mu(B_1) > \{0\}$ and $\mu(B_2) > \{0\}$. Since A is an atom of μ, it follows $\mu(A \backslash B_1) = \{0\}$. Since $B_2 \subseteq A \backslash B_1$ and μ is fuzzy, it results in $\mu(B_2) = \{0\}$, which is false. It results there is a unique $i_0 \in \{1, \ldots, n\}$ so that $\mu(B_{i_0}) > \{0\}$. Since $B_i \subseteq A \backslash B_{i_0}$, for every $i \in \{1, \ldots, n\} \backslash \{i_0\}$ and μ is fuzzy, it follows that $\mu(B_i) = \{0\}$, for every $i \in \{1, \ldots, n\} \backslash \{i_0\}$. So $\sum\limits_{i=1}^{n} |\mu(B_i)| \leq |\mu(A)|$. Since $\{B_i\}_{i=1}^n$ is an arbitrary partition of A, it results (3). □

Proposition 3.15. *Suppose $\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|)$ is a sn-exhaustive fuzzy set multifunction. Then for every $E \in \mathcal{P}(T)$ and every $\varepsilon > 0$, there is $A \in \mathcal{C}$ so that $A \subseteq E$ and $|\mu(B \backslash A)| < \varepsilon$, for all $B \in \mathcal{C}$, $A \subseteq B \subseteq E$.*

Proof. Suppose on the contrary that there exist $E_0 \in \mathcal{P}(T)$ and $\varepsilon > 0$ so that for all $A \in \mathcal{C}$, with $A \subseteq E_0$, there is $B_0 \in \mathcal{C}$ so that $A \subseteq B_0 \subseteq E_0$ and $|\mu(B_0 \backslash A)| \geq \varepsilon$. We construct by recurrence a sequence of mutual disjoint sets $(L_n) \subset \mathcal{C}$ such that $L_n \subseteq E_0$ for every $n \in \mathbb{N}$ and $|\mu(L_n)| \geq \varepsilon$.

Suppose we obtained $L_1, L_2, \ldots, L_n$ and let $K = \bigcup\limits_{i=1}^{n} L_i$. Obviously, $K \in \mathcal{C}$ and $K \subseteq E_0$. Then there is $B \in \mathcal{C}$ so that $K \subseteq B \subseteq E_0$ and $|\mu(B \backslash K)| \geq \varepsilon$. If we set $L_{n+1} = B \backslash K$, then we have $L_{n+1} \in \mathcal{C}$, $L_{n+1} \subseteq E_0$, $|\mu(L_{n+1})| \geq \varepsilon$ and $L_{n+1} \cap L_i = \emptyset$, for every $i \in \{1, \ldots, n\}$. Since μ is sn-exhaustive, we have $\lim\limits_{n \to \infty} |\mu(L_n)| = 0$, that is a contradiction. □

Definition 3.16. Suppose $|\cdot|$ is a monotone set-norm on $(\mathcal{P}_0(X), \leq)$ and let $\mathcal{R}$ be a non-empty subset of $\mathcal{P}_0(X)$.

(i) A net $(Y_i) \subset \mathcal{P}_0(X)$ is called *sn-Cauchy* if the net $(|Y_i|)$ is Cauchy (i.e., for every $\varepsilon > 0$, there is i_ε such that $||Y_i| - |Y_j|| < \varepsilon$, for every $i, j \geq i_\varepsilon$).

(ii) A net $(Y_i) \subset \mathcal{R}$ is called *sn-convergent* in $\mathcal{R}$ if there is a unique $Y_0 \in \mathcal{R}$ so that $\lim\limits_{i} |Y_i| = |Y_0|$. We denote this by $\lim\limits_{i} Y_i = Y_0$.

(iii) $\mathcal{R}$ is called *sn-complete* if every sn-Cauchy net of $\mathcal{R}$ is convergent in $\mathcal{R}$.

Example 3.17. Let $\mathcal{R} = \{[0, x] | x \in [0, +\infty)\}$. Then $\mathcal{R}$ is a sn-complete subspace of $(\mathcal{P}_0(\mathbb{R}), \subseteq, |\cdot|_s)$.

Theorem 3.18. *Suppose $|\cdot|$ is a monotone set-norm on $(\mathcal{P}_0(X), \leq)$, let $\mathcal{R}$ be a sn-complete subset of $(\mathcal{P}_0(X), \leq, |\cdot|)$ and let $\mu : \mathcal{C} \to \mathcal{R}$ be an sn-exhaustive subadditive fuzzy set multifunction. Then the following statements hold:*

(i) For every $E \in \mathcal{P}(T)$, there exists $\lim\limits_{\substack{A \in \mathcal{C} \\ A \subseteq E}} \mu(A) = \mu^*(E) \in \mathcal{R}$, where the net $(\mu(A))_{\substack{A \in \mathcal{C} \\ A \subseteq E}}$ is directed by the usual inclusion "$\subseteq$" of sets and the limit is in the sense of Definition 3.16-(ii). We obtain a set multifunction $\mu^* : \mathcal{P}(T) \to \mathcal{R}$.

(ii) $|\mu^*(A)| = |\mu(A)|, \forall A \in \mathcal{C}$.

(iii) $\forall E_1, E_2 \in \mathcal{P}(T),\ E_1 \subseteq E_2 \Rightarrow |\mu^*(E_1)| \leq |\mu^*(E_2)|$.

(iv) μ^* *is sn-exhaustive.*

(v) If μ *is non-atomic, then* μ^* *is non-atomic.*

Proof. (i) According to Proposition 3.15, for every $E \in \mathcal{P}(T)$ and $\varepsilon > 0$, there exists $A_0 \in \mathcal{C}$ so that $A_0 \subseteq E$ and for every $A \in \mathcal{C}$ with $A_0 \subseteq A \subseteq E$, we have

$$0 \leq |\mu(A)| - |\mu(A_0)| \leq |\mu(A \backslash A_0)| < \varepsilon.$$

So $(\mu(A))_{\substack{A\in\mathcal{C}\\ A\subseteq E}}$ is sn-Cauchy in $\mathcal{R}$ and since $\mathcal{R}$ is sn-complete, the net $(\mu(A))_{\substack{A\in\mathcal{C}\\ A\subseteq E}}$ is sn-convergent in $\mathcal{R}$.

(ii) Let $A \in \mathcal{C}$. For every $\varepsilon > 0$ there is $B_0 \in \mathcal{C}$ so that $B_0 \subseteq A$ and for every $B \in \mathcal{C}$ with $B_0 \subseteq B \subseteq A$ we have $||\mu(B)| - |\mu^*(A)|| < \varepsilon$. Particularly, $||\mu(A)| - |\mu^*(A)|| < \varepsilon$ for every $\varepsilon > 0$. Hence $|\mu^*(A)| = |\mu(A)|$.

For $A = \emptyset$ it results in $|\mu^*(\emptyset)| = |\mu(\emptyset)| = 0$ which implies that $\mu^*(\emptyset) = \{0\}$.

(iii) Let $E_1, E_2 \in \mathcal{P}(T)$ be so that $E_1 \subseteq E_2$. Consider an arbitrary $\varepsilon > 0$. Then there exists $A_1 \in \mathcal{C}$ so that $A_1 \subseteq E_1$ and for every $A \in \mathcal{C}$, with $A_1 \subseteq A \subseteq E_1$, we have $||\mu(A)| - |\mu^*(E_1)|| < \frac{\varepsilon}{2}$. Particularly we have

$$||\mu^*(E_1)| - |\mu(A_1)|| < \frac{\varepsilon}{2}. \tag{4}$$

Analogously there exists $A_2 \in \mathcal{C}$ so that $A_2 \subseteq E_2$ and for every $A \in \mathcal{C}$, with $A_2 \subseteq A \subseteq E_2$ we have

$$||\mu^*(E_2)| - |\mu(A)|| < \frac{\varepsilon}{2}.$$

Let $A_0 = A_1 \cup A_2 \in \mathcal{C}$. Then $A_2 \subseteq A_0 \subseteq E_2$ and we have

$$||\mu^*(E_2)| - |\mu(A_0)|| < \frac{\varepsilon}{2}. \tag{5}$$

Now, from (4) and (5) we obtain:

$$|\mu^*(E_1)| < |\mu(A_1)| + \frac{\varepsilon}{2} \leq |\mu(A_0)| + \frac{\varepsilon}{2} < |\mu^*(E_2)| + \varepsilon$$

for every $\varepsilon > 0$. Hence $|\mu^*(E_1)| \leq |\mu^*(E_2)|$.

(iv) Let $(E_n) \subset \mathcal{P}(T)$ be so that $E_n \cap E_m = \emptyset$, for every $m \neq n$. For every $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $A_n \in \mathcal{C}$ such that $A_n \subseteq E_n$ and $||\mu(A_n)| - |\mu^*(E_n)|| < \frac{\varepsilon}{2}$. Since $A_n \cap A_m = \emptyset$, for every $n \neq m$ and μ is sn-exhaustive, it results that there is $n_0 \in \mathbb{N}$ so that $|\mu(A_n)| < \frac{\varepsilon}{2}$, for every $n \geq n_0$. So we have:

$$|\mu^*(E_n)| \leq ||\mu^*(E_n)| - |\mu(A_n)|| + |\mu(A_n)| < \varepsilon,\ \ \forall n \geq n_0,$$

which proves that μ^* is sn-exhaustive.

(v) On the contrary suppose there is $E \in \mathcal{P}(T)$ an atom of μ^*. Then $\mu^*(E) > \{0\}$ and so $|\mu^*(E)| > 0$. It results that there is $A \in \mathcal{C}$ so that $A \subseteq E$ and $|\mu(A)| > 0$. Since $\mu(A) > \{0\}$ and μ is non-atomic, there exists $B \in \mathcal{C}$ such that $B \subseteq A$, $\mu(B) > \{0\}$ and $\mu(A \backslash B) > \{0\}$. Since $B \subseteq E$ and E is an atom of μ^*, it follows that $\mu^*(B) = \{0\}$ or $\mu^*(E \backslash B) = \{0\}$.

I. If $\mu^*(B) = \{0\}$, then by (ii) we have $|\mu(B)| = |\mu^*(B)| = 0$ and so $\mu(B) = \{0\}$ which is false.

II. If $\mu^*(E \backslash B) = \{0\}$, then by (ii) and (iii) we have:

$$0 < |\mu(A \backslash B)| = |\mu^*(A \backslash B)| \leq |\mu^*(E \backslash B)| = 0$$

that is a contradiction.
Consequently, μ^* is non-atomic. □

4. Conclusion

In this paper we presented properties of set-norm exhaustive set multifunctions and also some properties of atoms and pseudo-atoms of set-multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity. It would be interesting to see what non-atomicity becomes in the absence of fuzzyness of μ.

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APPROXIMATE FIXED POINT IN FUZZY NORMED SPACES FOR NONLINEAR MAPS

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Abstract. We define approximate fixed point in fuzzy norm spaces and prove the existence theorems, we also consider approximate pair constructive mapping and show its relation with approximate fuzzy fixed point.

1. Introduction

Chitra and Mordeson [8] defined fuzzy norm and thereafter the concept of fuzzy norm space has been introduced and generalized in different ways by Bag and Samanta in [2], [3], [4].

Definition 1.1. Let U be a linear space on $\mathbf{R}$. A function $N : U \times \mathbf{R} \to [0,1]$ is called a fuzzy norm if for every $x, u \in U$ and $c \in \mathbf{R}$ the following properties are satisfied.

(F_{N1}) $N(x,t) = 0$, for every $t \in \mathbf{R}^- \cup \{0\}$,

(F_{N2}) $N(x,t) = 1$, if and only if $x = 0$ for every $t \in \mathbf{R}^+$,

(F_{N3}) $N(cx,t) = N(x, \frac{t}{|c|})$ for every $c \neq 0$ and t $\in \mathbf{R}^+$,

(F_{N4}) $N(x+u, s+t) \geq min\{N(x,s), N(u,t)\}$, for every $s,t \in \mathbf{R}^+$,

(F_{N5}) the function $N(x,.)$ is nondecreasing on $\mathbf{R}$, and

$$\lim_{t\to\infty} N(x,t) = 1.$$

The pair (U,N) is called a fuzzy norm space. Sometimes, we need two additional conditions as follows :

(F_{N6}) $\forall t \in \mathbf{R}^+$ $N(x,t) > 0 \ \Rightarrow x = 0$.

(F_{N7}) The function $N(x,.)$ is continuous, for every $x \neq 0$, and is strictly increasing on the subset

$$\{t : \ 0 < N(x,t) < 1\}$$

is strictly increasing.

Let (U,N) be a fuzzy norm space. For all $\alpha \in (0,1)$, we define α norm on U as follows :

$$\|x\|_\alpha = \wedge\{t > 0 : N(x,t) \geq \alpha\} \ for \ every \ x \in U.$$

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Then $\{\|x\|_\alpha : \ \alpha \in (0,1]\}$ is an ascending family of normed on U and are called $\alpha - norm$ on U corresponding to the fuzzy norm N on U. Here are a lemma and example which will be used in this paper.

Lemma 1.2. [1] *Let (U,N) be a fuzzy norm space such that satisfy conditions F_{N6} and F_{N7}. Define the function $N' : U \times \boldsymbol{R} \to [0,1]$ as follows:*

$$N'(x,t) = \begin{cases} \vee\{\alpha \in (0,1) : \|x\|_\alpha \le t\} & (x,t) \ne (0,0) \\ \\ 0 & (x,t) = (0,0) \end{cases}$$

Then
a)N' is a fuzzy norm on U.
b) $N = N'$.

Lemma 1.3. [1] *Let (U,N) be a fuzzy norm space such that satisfy conditions F_{N6} and F_{N7}. and $\{x_n\} \subseteq U$, Then $\lim_{n\to\infty} N(x_n - x, t) = 1$ if and only if*

$$\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$$

for every $\alpha \in (0,1)$.

Note that the sequence $\{x_n\} \subseteq U$ converges if there exists $x \in U$ such that

$$\lim_{n\to\infty} N(x_n - x, t) = 1, for\ every\ t \in \boldsymbol{R}^+.$$

In this case x is called the limit of $\{x_n\}$.

Example 1.4. [1] Let V be a Real or Complex vector space and let N be defined on $V \times R$ as follows :

$$N(x,t) = \begin{cases} 1 & t > |x| \\ \\ 0 & t \le |x| \end{cases}$$

for all $x \in V$ and $t \in R$. Then (N,V) is a fuzzy norm space and the function N satisfies condition F_{N6} and $\|x\|_\alpha = |x|$, for every $\alpha \in (0,1)$.

Remark 1.5. Let (U,N) be a fuzzy norm space and $\{\|.\|_\alpha : \alpha \in (0,1)\}$ the set of all $\alpha-$norms on U. For two subsets A and B of U, we consider :

$$\delta(A,B) = \wedge\{\|x-y\|_\alpha : x \in A, y \in B,\ \alpha \in (0,1)\}.$$

Definition 1.6. Let (U,N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U and $T : A \cup B \to U$.

For a given $\epsilon > 0$, $x \in A \cup B$ is said to be a F^z-approximate fixed point for T if for some $\alpha \in (0,1)$

$$\|x - Tx\|_\alpha \le \delta(A,B) + \epsilon.$$

Remark 1.7. In the rest of the paper for a given $\epsilon > 0$, we will denote the set of all F^z-approximate fixed points of T, by

$$F_T^z(A,B) = \{x \in A \cup B : \|x - Tx\|_\alpha \le \delta(A,B) + \epsilon,\ for\ some\ \alpha \in (0,1)\}.$$

2. F^z-Approximate Fixed Point Maps

Proposition 2.1. *Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A, B be nonempty subsets of U and $T : A \cup B \to U$. If for $x \in A \cup B$ and $\alpha \in (0,1)$*

$$Lim_{n\to\infty}\|T^n x - T^{n+1}x\|_\alpha = \delta(A, B),$$

then there exists a F^z-approximate fixed point in $A \cup B$.

Proof. Suppose $x \in A \cup B$ and $\epsilon > 0$. Since

$$Lim_{n\to\infty}\|T^n x - T^{n+1}x\|_\alpha = \delta(A, B).$$

$$\exists N_0 > 0 \ s.t. \ \forall n \geq N_0 : \ \|T^n x - T^{n+1}x\|_\alpha < \delta(A, B) + \epsilon.$$

If $n = N_0$, then for some $\alpha \in (0,1)$

$$\|T^{N_0}x - T(T^{N_0}x)\|_\alpha < \delta(A, B) + \epsilon.$$

Therefore, $T^{N_0}x \in F^z_T(A, B)$ and $F^z_T(A, B) \neq \emptyset$. Hence there exists a F^z-approximate fixed point in $A \cup B$. □

Proposition 2.2. *Let (U, N) be a fuzzy norm space such that satisfy conditions F_{N6} and F_{N7} and $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U and $T : A \cup B \to U$. If for $x, y \in A \cup B$*

$$\|Tx - Ty\|_\alpha \leq m(\|x - y\|_\alpha) + l(\|x - Tx\|_\alpha + \|y - Ty\|_\alpha) + k\delta(A, B), \ \forall\alpha \in (0,1)$$

where $m, l, k \geq 0$ and $m + 2l + k < 1$. Then there exists a F^z-approximate fixed point in $A \cup B$.

Proof. If $x \in A \cup B$, then for some $\alpha \in (0,1)$ we have:

$$\|Tx - T^2x\|_\alpha \leq m(\|x - y\|_\alpha) + l(\|x - Tx\|_\alpha + \|Tx - T^2x\|_\alpha) + k\delta(A, B).$$

Therefore

$$\|Tx - T^2x\|_\alpha \leq \frac{m + l}{1 - l}\|x - Tx\|_\alpha + \frac{k}{1 - l}\delta(A, B).$$

Now if $p = \frac{m+l}{1-l}$, Then

$$\|Tx - T^2x\|_\alpha \leq p\|x - Tx\|_\alpha + (1 - p)\delta(A, B)$$

and $\|T^2x - T^3x\|_\alpha \leq p^2\|x - Tx\|_\alpha + (1 - p^2)\delta(A, B)$, hence,

$$\|T^n x - T^{n+1}x\|_\alpha \leq p^n\|x - Tx\|_\alpha + (1 - p^n)\delta(A, B).$$

since $p < 1$, for $\alpha \in (0,1)$, we have

$$\|T^n x - T^{n+1}x\|_\alpha \to \delta(A, B) \ as \ n \to \infty.$$

Therefore, by Proposition 2.1, there exists a F^z-approximate fixed point in $A \cup B$. □

Definition 2.3. Let (U, N) be a fuzzy norm space which satisfy conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U and $T : A \cup B \to U$. For some $\alpha \in (0,1)$ we define diameter $F^z_T(A, B)$ as follows :

$$diam(F^z_T(A, B)) = \vee\{\|x - y\|_\alpha : x, y \in F^z_T(A, B)\}.$$

Proposition 2.4. *Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U and $T : A \cup B \to U$. If there exists $k \in [0,1]$ such that $\|Tx - Ty\|_\alpha \le k\|x - y\|_\alpha$ for some $\alpha \in (0,1)$, then*

$$diam(F^z_T(A, B)) \le \frac{2\epsilon}{1-k} + \frac{2\delta(A, B)}{1-k}.$$

Proof. If $x, y \in A \cup B$, then for some $\alpha \in (0,1)$ we have

$$\begin{aligned}\|x - y\|_\alpha &\le \|x - Tx\|_\alpha + \|Tx - Ty\|_\alpha + \|Ty - y\|_\alpha \\ &\le \epsilon_1 + k\|x - y\|_\alpha + 2\delta(A, B) + \epsilon_2.\end{aligned}$$

By taking $\epsilon = Max\{\epsilon_1, \epsilon_2\}$, we have

$$\|x - y\|_\alpha \le \frac{2\epsilon}{1-k} + \frac{2\delta(A, B)}{1-k}.$$

Therefore $diam(F^z_T(A, B)) \le \frac{2\epsilon}{1-k} + \frac{2\delta(A,B)}{1-k}$. □

Definition 2.5. Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U and $T : A \cup B \to U$, $S : A \cup B \to U$. A point (x, y) in $A \times B$ is said a F^z-approximate fixed point for (T, S), if for some $\alpha \in (0,1)$ there exists $\epsilon > 0$ such that

$$\|(Tx, Sy)\|_\alpha \le \delta(A, B) + \epsilon.$$

Put :

$$F^z_{(T,S)}(A, B) = \{(x, y) \in A \times B : \|x - Tx)\|_\alpha \le \delta(A, B) + \epsilon \; for\ some\ \alpha \in (0,1)\}.$$

Proposition 2.6. *Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B be nonempty subsets of U and $T : A \cup B \to U$ and $S : A \cup B \to U$. If for $(x, y) \in A \times B$ and some $\alpha \in (0,1)$*

$$Lim_{n\to\infty}\|T^n x - S^n y\|_\alpha = \delta(A, B),$$

then there exists a F^z-approximate fixed point in $A \times B$.

Proof. For $\epsilon > 0$ and $\alpha \in (0,1)$, suppose $(x, y) \in A \times B$. Since

$$Lim_{n\to\infty}\|T^n x - S^n y\|_\alpha = \delta(A, B),$$

$$\exists N_0 > 0 \; s.t. \; \forall n \ge N_0 : \; \|T^n x - S^n y\|_\alpha < \delta(A, B) + \epsilon.$$

If $n = N_0$, then for some $\alpha \in (0,1)$

$$\|T(T^{N_0-1}x - S(S^{N_0-1}y)\|_\alpha < \delta(A,B) + \epsilon.$$

Now by takeing $x_0 = T^{N_0-1}(x)$ and $y_0 = S^{N_0-1}(y)$, we have

$$\|T(x_0) - S(y_0)\|_\alpha < \delta(A,B) + \epsilon,$$

hence $F^z_{(T,S)}(A,B) \neq \emptyset$. Therefore, there exists a F^z-approximate fixed point in $A \times B$. □

Proposition 2.7. *Let (U,N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B be nonempty subsets of U and $T : A \cup B \to U$ and $S : A \cup B \to U$. For any $(x,y) \in A \times B$ and for some $\alpha \in (0,1)$:*

$$\|Tx - Sy\|_\alpha \leq m(\|x-y\|_\alpha) + l(\|x - Tx\|_\alpha + \|y - Sy\|_\alpha) + k\delta(A,B),$$

where $m,l,k \geq 0$ and $m + 2l + k < 1$. If x is a F^z-approximate fixed point for T and y a F^z-approximate fixed point for S, then there exists a F^z-approximate fixed point in $A \times B$.

Proof. If $(x,y) \in A \times B$, then for some $\alpha \in (0,1)$ we have :

$$\|Tx - Sy\|_\alpha \leq m(\|x-y\|_\alpha) + l(\|x - Tx\|_\alpha + \|y - Sy\|_\alpha) + k\delta(A,B).$$

Therefore

$$\|Tx - S(Tx)\|_\alpha \leq \frac{m+l}{1-l}\|x - Tx\|_\alpha + \frac{k}{1-l}\delta(A,B).$$

Now if $p = \frac{m+l}{1-l}$, then

$$\|Tx - S(Tx)\|_\alpha \leq p\|x - Tx\|_\alpha + (1-p)\delta(A,B) \ (*)$$

and

$$\|Sy - T(Sy)\|_\alpha \leq p\|x - Tx\|_\alpha + (1-p)\delta(A,B) \ (**).$$

If x is a F^z-approximate fixed point for T, then there exists an $\epsilon > 0$ and by $(*)$ for some $\alpha \in (0,1)$:

$$\begin{aligned}\|Tx - S(Tx)\|_\alpha &\leq p\|x - Tx\|_\alpha + (1-p)\delta(A,B)\\ &\leq p(\delta(A,B) + \epsilon) + (1-p)\delta(A,B)\\ &= \delta(A,B) + p\epsilon\\ &< \delta(A,B) + \epsilon.\end{aligned}$$

Hence $(x, Tx) \in F^z_{(T,S)}(A,B)$, and
if y is a F^z-approximate fixed point for S, then there exists an $\epsilon > 0$ and by $(**)$ for some $\alpha \in (0,1)$:

$$\begin{aligned}\|Ty - T(Sy)\|_\alpha &\leq p\|y - Sy\|_\alpha + (1-p)\delta(A,B)\\ &\leq p(\delta(A,B) + \epsilon) + (1-p)\delta(A,B)\\ &= \delta(A,B) + p\epsilon\\ &< \delta(A,B) + \epsilon.\end{aligned}$$

Hence $(y, Sy) \in F^z_{(T,S)}(A, B)$. Therefore there exists a F^z-approximate fixed point in $A \times B$. □

Proposition 2.8. *Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B be nonempty subsets of U and $T : A \cup B \to U$. is a continuous map. For any $(x, y) \in A \times B$ and some $\alpha \in (0, 1)$*

$$\|Tx - Sy\|_\alpha \leq m(\|x - y\|_\alpha) + k\delta(A, B),$$

where $m, k \geq 0$ and $m + k = 1$ and the sequences $\{x_n\}, \{y_n\}$ be defined as follows :

$$x_{n+1} = Sy_n,\ y_{n+1} = Tx_n\ n \geq 0\ .$$

If $\{x_n\}$ has a convergent subsequence in A, then there exists a point $x_k \in A$ such that

$$\|x_k - Tx_k\|_\alpha = \delta(A, B)\ \alpha \in (0, 1).$$

Proof. We have

$$\begin{aligned} \|x_{n+1} - y_{n+1}\|_\alpha &= \|Tx_n - Sy_n\|_\alpha \\ &\leq m\|x_n - y_n\|_\alpha + (k)\delta(A, B) \\ &\leq ... \\ &\leq m^{n+1}\|x_0 - y_0\|_\alpha + (1 + m + m^2 + ... + m^n)(k)\delta(A, B) \end{aligned}$$

If $\{x_{n_p}\}_{p\geq 1}$ be convergent to $x_k \in A$, then

$$\|x_{n+1} - y_{n+1}\|_\alpha \leq m^{n_p+1}\|x_0 - y_0\|_\alpha + (1 + m + m^2 + ... + m^{n_p})(k)\delta(A, B).$$

Since T is continuous, we have,

$$\|x_{n_p+1} - Tx_{n_p}\| \to \frac{k}{1-m}\delta(A, B)\ \alpha \in (0, 1).$$

Therefore, $\|x_k - Tx_k\| = \delta(A, B)$. □

Example 2.9. Let us consider the linear space $U = C[0, 1]$, of all continuous real valued functions on [0,1] with the usual linear operations.
Consider two norms on $C[0, 1]$ defined by

$$\|x\|_0 = \{\int_{x_0}^1 (x(t))^2 dt\}^{\frac{1}{2}}$$

$$\|x\|_1 = \sup_{0\leq t\leq 1} |x(t)|.$$

Define two fuzzy norms N_1 and N_2 by

$$N_1(x, t) = \begin{cases} 1 & t > \|x\|_0 \\ 0 & t \leq \|x\|_0 \end{cases}$$

and

$$N_2(x,t) = \begin{cases} 1 & t > \|x\|_1 \\ \frac{1}{2} & \|x\|_0 \leq t \leq \|x\|_1 \\ 0 & t \leq \|x\|_0 \end{cases}$$

For the fuzzy norm N_1, its $\alpha-$norm $\|.\|_\alpha^1$ is given by $\|.\|_\alpha^1 = \|x\|_0 \ \forall \alpha \in (0,1)$.

For the fuzzy norm N_2, its $\alpha-$norm $\|.\|_\alpha^2$ is given by

$$\|x\|_\alpha = \begin{cases} \|x\|_0 & 0 < \alpha \leq \frac{1}{2} \\ \|x\|_1 & \frac{1}{2} < \alpha < 1 \end{cases}$$

Suppose A and B be nonempty subsets of U and $T : A \cup B \to U$. We consider $T(x) = x + 1$, $\forall x \in A \cup B$. It can easily be seen that for $\frac{1}{2} < \alpha < 1$, and $x, y \in X$.

$$\|Tx - Ty\|_\alpha \leq \frac{1}{2}\|x - y\|_\alpha$$

However, T has no any fixed point. But by the Proposition 2.2, for some $\epsilon > 0$, T has a F^z-approximate fixed point in $A \times B$. That is, there exists $x_0 \in U$ such that

$$\|T(x_0) - x_0\|_\alpha \leq \epsilon.$$

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WEAK AND STRONG DUALITY THEOREMS FOR FUZZY CONIC OPTIMIZATION PROBLEMS

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Abstract. The objective of this paper is to deal with the fuzzy conic programming problems. The aim here is to derive weak and strong duality theorems for a general fuzzy conic programming. Toward this end, The convexity-like concept of fuzzy mappings is introduced and then a specific ordering cone is established based on the parameterized representation of fuzzy numbers. Under this setting, duality theorems are extended from crisp conic optimization problems to fuzzy ones.

1. Introduction

Recently, convex fuzzy sets and convexity of fuzzy mappings have received considerable attention in dealing with mathematical treatment of fuzzy systems. In this regards, several types of convexity of fuzzy mappings and some important properties have been studied by many authors. Nanda et al. [14] and Syau [17] discussed the concept of convexity, invexity and B-preinvexity. Mishra et al. [13] introduced the concept of explicitly B-preinvex fuzzy mappings and Syau et al. [18] studied ϕ_1-convexity, Supp$-\phi_1-$convexity and Supp$-\phi_1-$qusiconvexity. Furthermore, the concept of pseudolinear and $\eta-$pseudolinear fuzzy mappings have been extended in [3, 9] by relaxing the definitions of pseudo-convex and pseudo-invex fuzzy mappings. Besides convex analysis which is a basic tool in the study of fuzzy convex optimization problems, weak and strong duality theorems for fuzzy optimization problems have also attracted a wide range of research. Rodder et al. [16] studied the duality of fuzzy linear programming problems and then this subject was extended by Bector et al. [1], Wu [20] and others [12, 15, 19]. Nowadays, Zhang et al. [22, 23] discussed the duality theory in fuzzy mathematical programming problems based on the convex fuzzy mappings.

Although, a lot of interesting explorations have been made in the study of duality theory in fuzzy optimization problems from different viewpoints, it seems that not much progress has been made in the aspect of extending the duality theorems for fuzzy conic optimization problems. Instead, here on the basis of the convexity-like concept of fuzzy mappings which is a more general than the convex one and an ordering cone defined by using parameterized representation of fuzzy numbers, the weak and strong duality theorems will be extended from crisp optimization

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problems to fuzzy ones. Notice that with the aid of the duality theorems and for the determination of solutions of the fuzzy primal problem [2, 8] we may consider the fuzzy dual problem that is possibly simpler than the fuzzy primal.

The present contribution is outlined as follows: Section 2 is devoted to give the definitions of fuzzy numbers and some results used later in the development of duality theorems in fuzzy environment. In Section 3 a boarder class of fuzzy optimization problems known as the conic convex programming is stated and primal and dual problems are established. Finally, Section 4 presents the fuzzy duality theorems and gives their detailed proofs based on convexity-like concept of fuzzy mappings.

2. **Preliminaries**

In the following, let a fuzzy number be defined as a fuzzy set $\widetilde{n} : \mathbb{R} \to [0,1]$ which is (see [4]): normal (i.e. there exist an element x_0 such that $\widetilde{n}(x_0) = 1$), fuzzy convex (i.e. $\widetilde{n}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{n}(x), \widetilde{n}(y)\}$, $\forall x, y \in \mathbb{R}$ and $\lambda \in [0,1]$), $\widetilde{n}$ is upper semi-continuous, $\texttt{Supp}(\widetilde{n})$ is bounded, where $\texttt{Supp}(\widetilde{n}) = \{x \in \mathbb{R};\ \widetilde{n}(x) > 0\}$ and its closure $\texttt{cl}(\texttt{Supp}(\widetilde{n}))$ is compact. Let $\mathbb{R}_F$ denote the family of fuzzy numbers. Also any real number $r \in \mathbb{R}$ can be regarded as a fuzzy number $\widetilde{r}$ defined by $\widetilde{r}(x) = 1$ if $x = r$, and $\widetilde{r}(x) = 0$ otherwise. Hence, the real number space $\mathbb{R}$ can be embedded in $\mathbb{R}_F$.

The α-level set of a fuzzy number $\widetilde{n} \in \mathbb{R}_F$, denoted by, $[\widetilde{n}]_\alpha$ is defined as a closed and bounded interval $[\underline{n}(\alpha), \overline{n}(\alpha)] = \{x \in \mathbb{R};\ \widetilde{n}(x) \geq \alpha\}$ if $0 < \alpha \leq 1$, and $[\underline{n}(\alpha), \overline{n}(\alpha)] = \texttt{cl}(\texttt{Supp}(\widetilde{n}))$ if $\alpha = 0$, where $\underline{n}(\alpha)$ and $\overline{n}(\alpha)$ denote respectively the left- and right-hand end points of $[\widetilde{n}]_\alpha$. As follows, a fuzzy number $\widetilde{n}$ can be identified by a parameterized triples $\{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) : \alpha \in [0,1]\}$. For $\widetilde{n}, \widetilde{m} \in \mathbb{R}_F$, represented respectively by $\{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) : \alpha \in [0,1]\}$ and $\{(\underline{m}(\alpha), \overline{m}(\alpha), \alpha) : \alpha \in [0,1]\}$, and $c \in \mathbb{R}$, the fuzzy operations can be defined by (see [4, 7])

$$\widetilde{n} + \widetilde{m} = \{(\underline{n}(\alpha) + \underline{m}(\alpha), \overline{n}(\alpha) + \overline{m}(\alpha), \alpha) : \ \alpha \in [0,1]\},$$
$$c\widetilde{n} = \{(c\underline{n}(\alpha), c\overline{n}(\alpha), \alpha) : \ \alpha \in [0,1]\}, \quad c \geq 0.$$

As is well known [6, 7], the space of fuzzy numbers does not constitute any linear space under operations mentioned above, that is, addition, subtraction and scalar multiplication derived from the usual extension principle.
Goetschel and Voxman [7] have embedded the space of fuzzy numbers in a topological vector space in their own manner different from the usual way.
In [7] the negative scalar multiplication of a fuzzy number $\widetilde{n} \in \mathbb{R}_F$ and a negative real number $c < 0$ is defined as

$$c\widetilde{n} = \{(c\overline{n}(\alpha), c\underline{n}(\alpha), \alpha) : \ \alpha \in [0,1]\},$$

where $\{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) : \ \alpha \in [0,1]\}$ being the parametric representation of $\widetilde{n}$.

From now on, the attention is restricted to the case of *bounded real-valued functions* $\underline{n}(\alpha)$, $\overline{n}(\alpha)$ on $[0,1]$. In view of this restriction, Goetschel and Voxman [7] pointed out that the family of parametric representations of members of $\mathbb{R}_F$ and the parametric representations of their negative scalar multiplications form subsets of the vector space

$$\aleph = \{\ \{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) :\ \alpha \in [0,1]\}\ ;\ \underline{n},\ \overline{n} : [0,1] \to \mathbb{R}\ \}.$$

Let the latter vector space be metricized by the metric

$$d(\widetilde{n}, \widetilde{m}) = \sup_{\alpha \in [0,1]} \max\{\ |\underline{n}(\alpha) - \underline{m}(\alpha)|\ ,\ |\overline{n}(\alpha) - \overline{m}(\alpha)|\}.$$

Follows from [13] the vector space $\aleph$ together with the metric d form a metric space $(\aleph, d)$. Let $\aleph_+ = \{\ \{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) :\ \alpha \in [0,1]\}\ ;\ \underline{n},\ \overline{n} : [0,1] \to \mathbb{R}^+\ \}$, where $\mathbb{R}^+$ denotes the set of all nonnegative real numbers.

Definition 2.1. [21] (i) Let C be a nonempty subset of a real linear space X. The set C is called a *cone* if $x \in C$ and $\lambda \geq 0$ then $\lambda x \in C$.

(ii) A cone C is said to be *convex* if $\lambda x + (1-\lambda)y \in C$ for $0 < \lambda < 1$ and any $x, y \in C$. A convex cone which characterizes the partial ordering on a real linear space is called an *ordering cone* or a *positive cone.*

(iii) Let X be the real linear space with an ordering cone C. The cone $C^* = \{l \in X^*;\ l(x) \geq 0,\ \forall x \in C\}$, is called the *dual cone* for C, where X^* denotes the dual space of X.

For the parameterized representation of fuzzy numbers, there exist some approaches [5, 7] that can be used to order (partially) the elements of $\mathbb{R}_F$. For instance, we can adopt the following approach:

Definition 2.2. For $\widetilde{n}, \widetilde{m} \in \mathbb{R}_F$ represented respectively by $\{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) :\ \alpha \in [0,1]\}$ and $\{(\underline{m}(\alpha), \overline{m}(\alpha), \alpha) :\ \alpha \in [0,1]\}$, a *partial order relation* on $\mathbb{R}_F$ defines $\widetilde{n} \preceq \widetilde{m}$ if and only if $\underline{n}(\alpha) \leq \underline{m}(\alpha)$ and $\overline{n}(\alpha) \leq \overline{m}(\alpha)$, $\forall \alpha \in [0,1]$ or if and only if $\widetilde{m} - \widetilde{n} \in \aleph_+$. Further, $\widetilde{n} \prec \widetilde{m}$ if and only if $\widetilde{m} - \widetilde{n} \in \aleph_+ \backslash \{\widetilde{0}\}$ and $\widetilde{n} \approx \widetilde{m}$ if and only if $\widetilde{n} - \widetilde{m} \in \aleph_+$ and $\widetilde{m} - \widetilde{n} \in \aleph_+$.

It should be noted that if the above partial ordering $\preceq$ is considered, then $\aleph_+$ is obviously a convex and ordering cone in $\mathbb{R}_F$.

Definition 2.3. Let S be a nonempty convex subset of the real linear space $\mathbb{R}^n$ and let $\mathbb{R}_F$ be the partially ordered fuzzy numbers space with the ordering cone $\aleph_+$. (i) A fuzzy mapping $\widetilde{f} : S \to \mathbb{R}_F$ is called *convex*, if $\lambda \widetilde{f}(x) + (1-\lambda)\widetilde{f}(y) - \widetilde{f}(\lambda x + (1-\lambda)y) \in \aleph_+$, for all $x, y \in S$ and $\lambda \in [0,1]$.

(ii) A fuzzy mapping $\widetilde{f} : S \to \mathbb{R}_F$ is called *convex-like*, if the fuzzy set $\widetilde{f}(S) + \aleph_+$ is convex.

Lemma 2.4. *Any convex fuzzy mapping $\widetilde{f} : S \to \mathbb{R}_F$ is also convex-like.*

Proof. It should be shown that the fuzzy set $\widetilde{f}(S) + \aleph_+$ is a convex set. For this purpose, choose arbitrary elements $\widetilde{y}_1, \widetilde{y}_2 \in \widetilde{f}(S) + \aleph_+$ and an arbitrary number $\lambda \in [0,1]$. Then, there are elements $x_1, x_2 \in S$ and $\widetilde{h}_1, \widetilde{h}_2 \in \aleph_+$ with $\widetilde{y}_1 \approx \widetilde{f}(x_1) + \widetilde{h}_1$, and $\widetilde{y}_2 \approx \widetilde{f}(x_2) + \widetilde{h}_2$. Thus, one obtains

$$\begin{aligned}&\lambda\widetilde{y}_1 + (1-\lambda)\widetilde{y}_2 \approx \lambda\widetilde{f}(x_1) + (1-\lambda)\widetilde{f}(x_2) + \lambda\widetilde{h}_1 + (1-\lambda)\widetilde{h}_2\\ &\in \{\widetilde{f}(\lambda x_1 + (1-\lambda)x_2);\ x_1, x_2 \in S,\ \lambda \in [0,1]\} + \aleph_+ + \lambda\aleph_+ + (1-\lambda)\aleph_+\\ &\text{(from convexity of } \widetilde{f})\\ &\approx \{\widetilde{f}(\lambda x_1 + (1-\lambda)x_2);\ x_1, x_2 \in S,\ \lambda \in [0,1]\} + \aleph_+,\end{aligned}$$

that is, $\lambda\widetilde{y}_1 + (1-\lambda)\widetilde{y}_2 \in \widetilde{f}(S) + \aleph_+$. Hence, the fuzzy set $\widetilde{f}(S) + \aleph_+$ is convex and the fuzzy mapping $\widetilde{f}$ is convex-like. □

Assumption 2.5. Before proceeding to present formally a fuzzy conic convex optimization problem, a number of assumptions are necessary.

A1. Let S be a nonempty subspace of the real linear space $\mathbb{R}$.

A2. Let $(\mathbb{R}_F, d)$ be a partially ordered fuzzy space with the ordering cone $\aleph_+$.

A3. Let $\widetilde{f} : S \to \mathbb{R}_F$ be a given fuzzy objective functional.

A4. Let $\widetilde{g} : S \to \mathbb{R}_F^n$ be a given fuzzy constraint mapping. Without loss of generality, n is taken to be 1.

A5. Let the composite fuzzy mapping $(\widetilde{f}, \widetilde{g}) : S \to \mathbb{R}_F^2$ be convex-like with respect to the product cone $\aleph_+^2$ in $\mathbb{R}_F^2$.

A6. Let the constraints set be given as $S_c = \{x \in S;\ \widetilde{g}(x) \in -\aleph_+\}$ which is assumed to be nonempty.

Remark that if these three things: the set S, fuzzy functional $\widetilde{f}$ and fuzzy mapping $\widetilde{g}$ are convex, then, composite fuzzy mapping $(\widetilde{f}, \widetilde{g}) : S \to \mathbb{R}_F^2$ is convex-like with respect to $\aleph_+^2$ in $\mathbb{R}_F^2$.

3. Fuzzy Conic Convex Programming

Now it is time to establish the general fuzzy conic convex programming, the so-called fuzzy constrained optimization(FCO) problem.

<u>**Problem FCO:**</u>

$$\begin{aligned}\text{minimize}\quad & \widetilde{f}(x)\\ \text{subject to}\quad & \widetilde{g}(x) \in -\aleph_+,\\ & x \in S.\end{aligned}$$

In optimization theory, Problem FCO is also called *primal* problem.

In the remainder of this section two theorems which are known [11, 21] as *Eidelheit separation theorem* and *separation theorem*, must be taken into account because of furnishing the proofs of the main results in this study.

Theorem 3.1. *(Eidelheit separation) Let S and W be nonempty convex subsets of a real topological linear space X with $int(S) \neq \emptyset$. Then, $int(S) \cap W = \emptyset$ if and only if there are a continuous linear functional $l \in X^* \backslash \{0_{X^*}\}$ and a real number ζ with*

$$l(s) \le \zeta \le l(w), \quad \forall s \in S,\ \forall w \in W,\ \textit{and}\ l(s) < \zeta, \quad \forall s \in int(S).$$

Theorem 3.2. *(Separation) Let S be a nonempty convex and closed subset of a real locally convex space X. Then, $x \in X \backslash S$ if and only if there is a continuous linear functional $l \in X^* \backslash \{0_{X^*}\}$ with $l(x) < \inf_{s \in S} l(s)$.*

Recall that the ordering cone $\aleph_+$ is closed. As a first result, next lemma states Problem FCO is equivalent to the following fuzzy unconstrained optimization(FUCO) problem.

Problem FUCO:

$$\min_{x \in S} \quad \sup_{l \in \aleph_+^*} \widetilde{f}(x) + l(\widetilde{g}(x))$$

where $\aleph_+^*$ denotes the dual cone of $\aleph_+$.

Lemma 3.3. *Suppose the assumptions A1-A6 are fulfilled. Then $\bar{x}$ is a minimal solution of the Problem FCO if and only if $\bar{x}$ is a minimal solution of Problem FUCO. In this case the extremal values of both optimization problems are fuzzy equal.*

Proof. It is firstly assumed that $\bar{x} \in S_c$ is a minimal solution of the Problem FCO, that is, $\widetilde{f}$ attains its minimum at $\bar{x}$ on S_c. For every $x \in S$ with $\widetilde{g}(x) \in -\aleph_+$, it holds that

$$l(\widetilde{g}(x)) \leq 0, \quad \forall l \in \aleph_+^*,$$

therefore, it is guaranteed that $\sup_{l \in \aleph_+^*} l(\widetilde{g}(x)) = 0$.
Using Theorem 3.2 and the fact that $\aleph_+$ is convex and closed, for any $x \in S$ with $\widetilde{g}(x) \notin -\aleph_+$ there exists an $\bar{l} \in \aleph_+^* \backslash \{0_{\mathbb{R}}\}$ such that $\bar{l}(\widetilde{g}(x)) > 0$, resulting in $\sup_{l \in \aleph_+^*} l(\widetilde{g}(x)) = \infty$.
Furthermore, one can easily verify that for every $x \in S$

$$\begin{aligned}
\sup_{l \in \aleph_+^*} \widetilde{f}(\bar{x}) + l(\widetilde{g}(\bar{x})) &\approx \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{f}(\bar{x}) + \widetilde{l}(\widetilde{g}(\bar{x})) \\
&\approx \widetilde{f}(\bar{x}) + \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{l}(\widetilde{g}(\bar{x})) \\
&\approx \widetilde{f}(\bar{x}) \\
&\preceq \widetilde{f}(\bar{x}) + \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{l}(\widetilde{g}(x)) \\
&\preceq \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{f}(x) + \widetilde{l}(\widetilde{g}(x)) \approx \sup_{l \in \aleph_+^*} \widetilde{f}(x) + l(\widetilde{g}(x)).
\end{aligned}$$

Hence, $\bar{x} \in S_c$ is also a minimal solution of Problem FUCO.
To prove the converse assertion, suppose that $\bar{x} = Arg \min_{x \in S} \{\sup_{l \in \aleph_+^*} \widetilde{f}(x) + l(\widetilde{g}(x))\}$. If $\widetilde{g}(\bar{x}) \notin -\aleph_+$ then, by the same arguments as used above one gets $\sup_{l \in \aleph_+^*} l(\widetilde{g}(\bar{x})) = \infty$, that obviously contradicts the assumption of the solvability of Problem FUCO. In this case, it results in $\sup_{l \in \aleph_+^*} l(\widetilde{g}(\bar{x})) = 0$.

Now it can be shown that for all $x \in S_c$

$$\begin{aligned} \widetilde{f}(\bar{x}) \approx \widetilde{f}(\bar{x}) + \sup_{l \in \aleph_+^*} l(\widetilde{g}(\bar{x})) &\approx \widetilde{f}(\bar{x}) + \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{l}(\widetilde{g}(\bar{x})) \\ &\approx \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{f}(\bar{x}) + \widetilde{l}(\widetilde{g}(\bar{x})) \\ &\preceq \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{f}(x) + \widetilde{l}(\widetilde{g}(x)) \\ &\approx \widetilde{f}(x) + \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{l}(\widetilde{g}(x)) \approx \widetilde{f}(x). \end{aligned}$$

This simply means that $\bar{x} \in S_c$ is a minimizer of $\widetilde{f}$ on S_c and the axiom is satisfied. □

Now it is time to associate another problem to the primal Problem FCO. This new problem is produced by a two-stage process from Problem FUCO. The first stage is carried out by exchanging **min** and **sup** and in the second stage **min** and **sup** are then replaced by **inf** and **max**, respectively. The resulted optimization problem, the so-called *dual* problem reads:

Problem FDUCO:

$$\max_{l \in \aleph_+^*} \quad \inf_{x \in S} \widetilde{f}(x) + l(\widetilde{g}(x)).$$

The equivalency of Problem FCO and Problem FUCO demonstrated by Lemma 3.3 may be investigated again between Problem FDUCO and the following one.

Problem FDCO:

$$\begin{aligned} \text{maximize} \quad & \widetilde{\lambda} \\ \text{subject to} \quad & \widetilde{f}(x) + l(\widetilde{g}(x)) \succeq \widetilde{\lambda}, \quad \forall x \in S, \\ & \widetilde{\lambda} \in \mathbb{R}_F, \; l \in \aleph_+^*. \end{aligned}$$

Analogous to the pervious case, the equivalence may be stated as: if $\bar{l} \in \aleph_+^*$ is a maximal solution of the dual Problem FDUCO with the maximal value $\bar{\widetilde{\lambda}}$, then $(\bar{\widetilde{\lambda}}, \bar{l})$ is a maximal solution of Problem FDCO. Conversely, for every maximal solution $(\bar{\widetilde{\lambda}}, \bar{l})$ of the Problem FDCO, $\bar{l}$ is a maximal solution of Problem FDUCO with the maximal value $\bar{\widetilde{\lambda}}$.

4. Weak and Strong Duality Theorems

The objective of this section is to derive the relationships between the primal Problem FCO and the dual Problem FDCO by means of the so-called *weak duality theorem* and *strong duality theorem*. The major result that will be shown later says that the primal and dual problems are equivalent.

Theorem 4.1. *(Weak duality) Suppose that assumptions A1-A6 are satisfied. Let* $\hat{x} \in S_c$ *and* $\widehat{l} \in \aleph_+^*$ *with the maximal value* $\widetilde{\widehat{\lambda}} \approx \inf_{x\in S} \widetilde{f}(x) + \widehat{l}(\widetilde{g}(x))$ *be respectively the feasible solutions of primal Problem FCO and dual Problem FDCO. Then the objective value of dual Problem FDCO is always fuzzy less than or fuzzy equal to the objective value of primal Problem FCO, that is:*

$$\inf_{x\in S} \widetilde{f}(x) + \widehat{l}(\widetilde{g}(x)) \preceq \widetilde{f}(\hat{x}).$$

Proof. Using the fact that $\widetilde{g}(\hat{x}) \in -\aleph_+$, one completes the proof in a straightforward manner by showing that for any $\hat{x} \in S_c$ and $\widehat{l} \in \aleph_+^*$

$$\inf_{x\in S} \widetilde{f}(x) + \widehat{l}(\widetilde{g}(x)) \preceq \widetilde{f}(\hat{x}) + \widehat{l}(\widetilde{g}(\hat{x})) \preceq \widetilde{f}(\hat{x}).$$

This holds true, since $\widetilde{g}(\hat{x}) \in -\aleph_+$ leads to $\widehat{l}(\widetilde{g}(x)) \leq 0$. □

Now, in order to investigate the strong duality theorem, it needs beforehand to provide a sufficient level of detail in describing the concepts used within the study. (See [10])

Definition 4.2. (i) A fuzzy subset $\widetilde{A} \subset \widetilde{B} = \{(x, \mu_{\widetilde{B}}(x));\ x \in \mathbb{R},\ 0 \leq \mu_{\widetilde{B}}(x) \leq 1\}$ is a *fuzzy neighborhood* of $x_0 \in \widetilde{B}$ if and only if $\mu_{\widetilde{A}}(x_1) - \epsilon < \mu_{\widetilde{B}}(x_0) < \mu_{\widetilde{A}}(x_2) + \epsilon$, for all $x_1, x_2 \in \widetilde{B}$ and $0 < \epsilon < 1$.

(ii) Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy sets such that $\widetilde{A} \subseteq \widetilde{B}$. Then, the fuzzy set $\widetilde{A}$ is said to be *fuzzy interior* to $\widetilde{B}$ if and only if $\widetilde{A}$ is a fuzzy neighborhood of $\widetilde{B}$.

(iii) The *interior of* $\widetilde{B}$ is the union of all fuzzy interior subsets of $\widetilde{B}$, that is, the interior of $\widetilde{B}$ denoted by $int(\widetilde{B})$ is the largest open fuzzy set contained in $\widetilde{B}$.

Definition 4.3. Recall the assumption A6 which states $S_c = \{x \in S;\ \widetilde{g}(x) \in -\aleph_+\}$. It is called that *the generalized Slater condition* holds true if there exists a vector $\hat{x} \in S_c$ such that $\widetilde{g}(\hat{x}) \in -int(\aleph_+)$.

Theorem 4.4. *(Strong duality) Suppose that the assumptions A1-A6 are fulfilled. Furthermore, assume that the primal Problem FCO is solvable and the generalized Slater condition is satisfied. Then the dual Problem FDCO is also solvable and there is no gap between the primal Problem FCO and the dual Problem FDCO, i.e. the extremal values of the two problems are fuzzy equal.*

Proof. The procedure of the proof starts with constructing a fuzzy set as follows:

$\widetilde{M} = \{(\widetilde{f}(x) + \widetilde{F},\ \widetilde{g}(x) + \widetilde{G}) \in \mathbb{R}_F^2;\ x \in S,\ \widetilde{F} \succeq \widetilde{0},\ \widetilde{G} \in \aleph_+\} = (\widetilde{f}, \widetilde{g})(S) + \aleph_+^2.$

As follows from assumptions A1-A6, one can observe that the composite fuzzy mapping $(\widetilde{f}, \widetilde{g}) : S \to \mathbb{R}_F^2$ is convex-like and therefore the set $\widetilde{M}$ is convex. Since $int(\aleph_+) \neq \emptyset$, this implies that the set $\widetilde{M}$ has a nonempty interior $int(\widetilde{M})$ as well. With respect to the solvability of the primal Problem FCO, there exists a vector $\bar{x} \in S_c$ such that $\widetilde{f}(\bar{x}) \preceq \widetilde{f}(x),\ \forall x \in S_c$. This shows that $(\widetilde{f}(\bar{x}), \widetilde{0}) \notin int(\widetilde{M})$, and therefore $int(\widetilde{M}) \cap \{(\widetilde{f}(\bar{x}), \widetilde{0})\} = \emptyset$. Regarding the latter attainment and

Theorem 3.1, there exist a real number ζ and a composite continuous linear functional $(l_1, l_2) \in \mathbb{R}_F^{*2}$ with $(l_1, l_2) \neq (0,0)$ so that $(l_1, l_2) \circ (\widetilde{\widetilde{f}}, \widetilde{\widetilde{g}}) \geq \zeta \geq (l_1, l_2) \circ (\widetilde{f}(\bar{x}), \widetilde{0})$, $\forall (\widetilde{\widetilde{f}}, \widetilde{\widetilde{g}}) \in int(\widetilde{M})$, or

$$l_1(\widetilde{\widetilde{f}}) + l_2(\widetilde{\widetilde{g}}) \geq \zeta \geq l_1(\widetilde{f}(\bar{x})) + 0, \quad \forall (\widetilde{\widetilde{f}}, \widetilde{\widetilde{g}}) \in int(\widetilde{M}). \tag{1}$$

Recall that any convex subset of a normed space whose interior set is nonempty, is contained in the closure of that its interior. According to this assertion one may conclude from the inequality (1)

$$l_1(\widetilde{f}(x) + \widetilde{F}) + l_2(\widetilde{g}(x) + \widetilde{G}) \geq \zeta \geq l_1(\widetilde{f}(\bar{x})), \quad \forall x \in S,\ \widetilde{F} \succeq \widetilde{0},\ \widetilde{G} \in \aleph_+ . \tag{2}$$

In the case that $x = \bar{x}$ and $\widetilde{F} \approx \widetilde{0}$, it follows immediately from the inequality (2) that

$$l_2(\widetilde{G}) \geq -l_2(\widetilde{g}(\bar{x})), \quad \forall \widetilde{G} \in \aleph_+ . \tag{3}$$

Needless to say that $l_2 \in \aleph_+^*$. Further, imposing $\widetilde{G} \approx \widetilde{0}$ in inequality (3) results in $l_2(\widetilde{g}(\bar{x})) \geq 0$. On the other hand, from $\widetilde{g}(\bar{x}) \in -\aleph_+$ and $l_2 \in \aleph_+^*$ one gets $l_2(\widetilde{g}(\bar{x})) \leq 0$, which leads to $l_2(\widetilde{g}(\bar{x})) = 0$.

Take $x = \bar{x}$ and $\widetilde{G} \approx \widetilde{0}$. In this regard, the inequality (2) becomes $l_1(\widetilde{F}) \geq 0,\ \forall \widetilde{F} \succeq \widetilde{0}$, which implies that $l_1 \geq 0$. The aim is to show that $l_1 > 0$. Suppose it is not the case, that is, $l_1 = 0$, then the inequality (2) with $\widetilde{G} \approx \widetilde{0}$ becomes $l_2(\widetilde{g}(x)) \geq 0,\ \forall x \in S$. In virtue of the latter inequality and the generalized Slater condition, there is one $\hat{x} \in S$ so that $\widetilde{g}(\hat{x}) \in -int(\aleph_+)$. Thus $l_2(\widetilde{g}(\hat{x})) = 0$. This gives rise to $l_2 = 0$. By contradiction, let $l_2 \neq 0$, that is, there is one $\widetilde{G} \in \mathbb{R}_F$ such that $l_2(\widetilde{G}) > 0$. Then, it is not hard to see that

$$l_2(\lambda\widetilde{G} + (1-\lambda)\widetilde{g}(\hat{x})) > 0, \quad \forall \lambda \in [0,1]. \tag{4}$$

In this case, $\widetilde{g}(\hat{x}) \in -int(\aleph_+)$ provides the existence of a $\bar{\lambda} \in (0,1)$ such that

$$\lambda\widetilde{G} + (1-\lambda)\widetilde{g}(\hat{x}) \in -int(\aleph_+), \quad \forall \lambda \in [0, \bar{\lambda}].$$

This implies that $l_2(\lambda\widetilde{G} + (1-\lambda)\widetilde{g}(\hat{x})) \leq 0,\ \forall \lambda \in [0, \bar{\lambda}]$, which is contradict the inequality (4).

Note that the above argument shows that if $l_1 = 0$, this may leads to $l_2 = 0$. This contradicts $(l_1, l_2) \neq (0,0)$. Thereby, it must be $l_1 \neq 0$ and therefore $l_1 > 0$.

Now, by taking $\widetilde{F} \approx \widetilde{0}$ and $\widetilde{G} \approx \widetilde{0}$, one may obtain from the inequality (2) that

$$l_1(\widetilde{f}(x)) + l_2(\widetilde{g}(x)) \geq l_1(\widetilde{f}(\bar{x})), \quad \forall x \in S,$$

equivalently

$$\widetilde{f}(x) + {l_1}^{-1} l_2(\widetilde{g}(x)) \geq \widetilde{f}(\bar{x}), \quad \forall x \in S.$$

The subsequence results follow directly from $\bar{l} = {l_1}^{-1} l_2 \in \aleph_+^*$ with $\bar{l}(\widetilde{g}(\bar{x})) = 0$. That is, to say,

$$\widetilde{f}(x) + \bar{l}(\widetilde{g}(x)) \succeq \widetilde{f}(\bar{x}) + \bar{l}(\widetilde{g}(\bar{x})), \quad \forall x \in S.$$

Hence,

$$\widetilde{f}(\bar{x}) \approx \widetilde{f}(\bar{x}) + \bar{l}(\widetilde{g}(\bar{x})) \approx \inf_{x \in S} \widetilde{f}(x) + \bar{l}(\widetilde{g}(x)),$$

where by Theorem 4.1, $\bar{l} \in \aleph_+^*$ is a maximal solution of the dual Problem FDCO. It is obvious from the above fuzzy equality that the extremal values of the primal Problem FCO and the dual Problem FDCO are fuzzy equal. □

5. **Conclusion**

The aim of the paper is to present an extension of the classical duality theorems known as weak and strong duality theorems for conic programming problems to those arising for fuzzy ones. The key ideas are the convexity-like concept of fuzzy mappings and the treatment of the ordering cone $\aleph_+$ established upon the parameterized representation of fuzzy numbers.

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PERSIAN TRANSLATION OF ABSTRACTS

A CONSTRAINED SOLID TSP IN FUZZY ENVIRONMENT: TWO HEURISTIC APPROACHES

C. CHANGDAR, M. K. MAITI AND M. MAITI

یک TSP ثابت مقید در محیط فازی: دو روش ابتکاری

چکیده. مسئله فروشنده دوره گرد ثابت (STSP) مسئله فروشنده دوره گردی است که در سفر خود از شهری به شهر دیگر با استفاده از وسایل نقلیه مختلف از تمام شهرها تنها یک بار دیدن می کند. هزینه ها و عوامل موثر محیطی برای سفر بین شهرها توسط وسایل نقلیه مختلف متفاوت است. هدف مسئله یافتن یک سفر کامل با کمترین هزینه وکمترین خسارت وارده به محیط است. برای حل مسئله یک الگوریتم بهینه سازی کلن مورچه (ACO) توسعه داده شده است. اجرای الگوریتم مسئله با الگوریتم محاسبه نرم دیگری مقایسه شده است، الگوریتم تکاملی (GA) . مسائل وهمچنین هزینه های فازی بطور قطعی حل می شوند. برای هزینه فازی و عوامل موثر محیطی، تابع هزینه ومحدودیتهای محیطی فازی می شوند. چون بهینه سازی یک تابع هدف فازی خوش تعریف نیست، روش احتمال فازی بکاربرده شده تا تصمیم مطلوب بدست آید. برای امتحان کارآیی الگوریتم، مسئله تنها با درنظر گرفتن یک نقلیه ونادیده گرفتن محدودیت موثر محیطی، یعنی یک TSP دو بعدی کلاسیک (انتخاب مجموعه های داده استاندارد TSPLIB برای حل مسئله) حل می شود. برای توضیح ، مثالهای عددی بسیاری آورده شده است .

A COGNITIVE STYLE AND AGGREGATION OPERATOR MODEL: A LINGUISTIC APPROACH FOR CLASSIFICATION AND SELECTION OF THE AGGREGATION OPERATORS

K. K. F. YUEN

روش ادراکی ومدل عملگر تراکم: روش زبان شناختی برای رده بندی و انتخاب عملگرهای تراکم

چکیده. عملگرهای تراکم (AOs) توسط بسیاری از محققین مطالعه شده اند. با وجود آنکه عملگرهای تراکم بسیاری پیشنهاد شده اند، هنوز کمبود روشهای رده بندی رسته های عملگر تراکم و انتخاب AO مناسب از بین AO های پیشنهادی وجود دارد. در این تحقیق هر AO می تواند به عنوان یک روش ادراکی منحصر بفرد در نظر گرفته شود. برای بررسی نگاشت رابطه بین عملگرهای تراکم وروشهای ادراکی بیان شده توسط حالتهای تصمیم، یک روش ادراکی وعملگر تراکم پیشنهاد شده است. برای CSAO : CSAO-۱ ، CSAO-۲ ودو استراتژی انتخاب بر مبنای CSAO-۱ و CSAO-۲ ، چهار الگوریتم پیشنهاد شده است. مثالهای عددی چگونگی انتخاب عملگرهای تراکم بر اساس شرایط تصمیم با انتخاب استراتژی های CSAO-۱ وCSAO-۲ را مشخص می کنند. مدل CSAO می تواند با انتخاب مسائل مناسب از عملگرهای تراکم با در نظر گرفتن روشهای ادراکی تصمیم گیرنده ها، برای سیستمهای تصمیم گیری به کار گرفته شود.

FUZZY GOAL PROGRAMMING TECHNIQUE TO SOLVE MULTIOBJECTIVE TRANSPORTATION PROBLEMS WITH SOME NON-LINEAR MEMBERSHIP FUNCTIONS

M. ZANGIABADI AND H. R. MALEKI

روش برنامه ریزی فازی با توابع عضویت غیرخطی برای حل مسائل حمل و نقل چند هدفه

چکیده. مساله حمل و نقل چند هدفه ی خطی نوع خاصی از مسائل کمینه سازی برداری است که در آن قیود همگی از نوع تساوی بوده و توابع هدف معمولاً با یکدیگر در تضاد هستند. این مقاله کاربردی از برنامه ریزی آرمانی فازی را برای مساله ی حمل و نقل چند هدفه ارائه می دهد. در این مقاله، ما از توابع عضویت غیر خطی مثل تابع عضویت هذلولوی و تابع عضویت نمایی، برای حل مساله ی حمل و نقل استفاده می کنیم. این روش یک جواب بهینه ی توافقی برای مساله پیدا می کند. نتایج بدست آمده با نتایج حاصل از روش آرمانی فازی با تابع عضویت خطی مقایسه می شوند. با حل مثال های عددی، الگوریتم ارائه شده توضیح داده می شود.

MINIMIZATION OF DETERMINISTIC FINITE AUTOMATA WITH VAGUE (FINAL) STATES AND INTUITIONISTIC FUZZY (FINAL) STATES

A. CHOUBEY AND K. M. RAVI

کمینه سازی قطعی ماشین خودکار متناهی با وضعیت های (نهایی) مبهم ووضعیت های (نهایی) فازی شهودی

چکیده. دراین مقاله، روابط بین مقادیر عضویت زبانهای فازی تعمیم یافته مانند زبان فازی شهودی، زبان فازی بازده- مقدار وزبان مبهم مطالعه شده اند. این روابط به مطالعه ی خواص یک زبان هنگامی که خواص دیگر مشخص می باشند کمک خواهد کرد. بعلاوه دراین مقاله، برای هر زبان منظم مبهم، شناخته شده توسط یک رباط (متناهی) با وضعیت های (نهایی) مبهم وجود یک رباط متناهی کمینه با وضعیت های(نهایی) مبهم نشان داده شده است. نهایتاً، برای کمینه سازی رباط متناهی با وضعیت های(نهایی) مبهم یک الگوریتم کارا ارائه شده است. به طور مشابه، برای زبان منظم فازی شهودی نیز می توان نشان داد. اینها ممکن است ضمن مطالعه ی تحلیل لغوی، تصمیم گیری وغیره در فهم بهتر نقش رباط متناهی با وضعیت های (نهایی) مبهم یا رباط متناهی با وضعیت های (نهایی) فازی شهودی کمک کند.

ON THE DIAGRAM OF ONE TYPE MODAL OPERATORS ON INTUITIONISTIC FUZZY SETS: LAST EXPANDING WITH $Z_{\alpha,\beta}^{\omega,\theta}$

G. CUVALCIOGLU

دیاگرام عملگرهای شرطی نوع یک روی مجموعه های فازی شهودی: آخرین گسترش

چکیده. عملگر شرطی فازی شهودی فازی توسط توسط Atanassov در سال ۱۹۹۹ در [۳] تعریف شد. درسال ۲۰۰۱در[۴] ، او تعمیمی از این عملگرهای شرطی را معرفی نمود. بعد از آن Dencheva درسال ۲۰۰۴ در[۴] دومین گسترش از عملگرها را تعریف کرد. در سال ۲۰۰۶ سومین گسترش از آنها در [۶] توسط Atanassov تعریف شد . در سال ۲۰۰۷ نویسنده در [۱۱] عملگر جدیدی را روی مجموعه های فازی شهودی معرفی نمود که گسترشی از عملگرهای Atanassov وDencheva بود. در همان سال Atanassov عملگری را تعریف نمود که تعمیمی از تمام عملگرهایی بود که تا سال ۲۰۰۷ تعریف شده بودند. دیاگرام عملگرهای شرطی نوع یک روی مجموعه های فازی شهودی ابتدا در سال ۲۰۰۷ توسط Atanassovدر [۱۰] معرفی گردید. درسال ۲۰۰۷ Atanassov جامع ترین عملگر را تعریف نمود ودرسال ۲۰۱۰ نویسنده دیاگرام عملگرهای شرطی از نوع یک روی مجموعه های فازی شهودی را با عملگر $Z_{\alpha,\beta}^{\omega,\theta}$ گسترش داد. برخی از روابط بین این عملگرها توسط چندین محقق در [۸] -[۵] ،[۱۱] ، و[۱۶]–[۱۴] مورد بررسی قرار گرفت. هدف این مقاله گسترش دیاگرام عملگرهای شرطی نوع یک روی مجموعه های فازی شهودی است برای این منظور، عملگر جدید شرطی $Z_{\alpha,\beta}^{\omega,\theta}$ را روی مجموعه های فازی شهودی تعریف می کنیم. نشان می دهیم که این عملگر تعمیمی از عملگرهای $Z_{\alpha,\beta}^{\omega,\theta}$ ، $E_{\alpha,\beta}$ ، $\boxtimes_{\alpha,\beta}$ ، $\boxplus_{\alpha,\beta}$ می باشد.

FUZZY INTEGRO-DIFFERENTIAL EQUATIONS: DISCRETE SOLUTION AND ERROR ESTIMATION

M. ZEINALI, S. SHAHMORAD AND K. MIRNIA

معادلات انتگرال- دیفرانسیل فازی: جواب گسسته و تخمین خطا

چکیده. این مقاله نتایج وجود و یکتایی جواب را برای معادلات انتگرال- دیفرانسیل فازی مرتبه اول مورد بررسی قرار می دهد. سپس نتایج عددی و کران خطا براساس قواعد انتگرالگیری عددی مستطیلی چپ، ذوزنقه ای و یک ترکیبی از این دوروش بدست می آید. در پایان مثالی ارائه می گردد تا کارایی این روش ها را روشن نماید.

SET-NORM EXHAUSTIVE SET MULTIFUNCTIONS

A. CROITORU AND A. GAVRILUT

نرم مجموعه ای کامل توابع چندگانه مجموعه ای

چکیده. در این مقاله برخی از خواص نرم مجموعه ای کامل توابع چندگانه مجموعه ای و همچنین اتمها وشبه اتمهای مجموعه ی توابع چند گانه که مقادیر خود را در خانواده ای از زیر مجموعه های غیرتهی از نیمگروههای جابجایی واحددار انتخاب می کنند، ارائه می دهیم.

APPROXIMATE FIXED POINT IN FUZZY NORMED SPACES FOR NONLINEAR MAPS

S. A. M. MOHSENIALHOSSEINI, H. MAZAHERI AND M. A. DEHGHAN

تقریب نقطه ثابت در فضا های نرمدار فازی برای نگاشت های غیر خطی

چکیده. تقریب نقطه ثابت در فضا های نرمدار فازی را تعریف می کنیم و قضایای وجودی را ثابت می کنیم. تقریب زوج تقریبی را برای نگاشت های انقباضی بدست می آوریم و ارتباط آنرا با تقریب نقطه ثابت فازی بدست می آوریم.

WEAK AND STRONG DUALITY THEOREMS FOR FUZZY CONIC OPTIMIZATION PROBLEMS

B. FARHADINIA AND A. V. KAMYAD

قضایای دوگانی قوی و ضعیف برای مسایل بهینه سازی مخروطی فازی

چکیده. در این مقاله مسایل بهینه سازی مخروطی فازی مورد توجه قرار گرفته است، بویژه قضایای دوگانی قوی و ضعیف برای مسایل بهینه سازی مخروطی فازی در فرم جامع آنها مورد بررسی قرار خواهند گرفت. برای این منظور مفهوم تحدب- مانند نگاشت های فازی معرفی می گردد سپس مخروطی مرتب بر مبنای نمایش پارامتری اعداد فازی پایه گذاری می گردد. بر این اساس، قضایای دوگانی از حالت کلاسیک به حالت فازی برای مسایل بهینه سازی مخروطی توسیع داده می شوند.

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